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Algebra I Teacher's Edition

Andrew McFarland
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CHAPTER 1 TE Equations and Functions

CHAPTER OUTLINE

- 1.1 VARIABLE EXPRESSIONS
 - 1.2 ORDER OF OPERATIONS
 - 1.3 PATTERNS AND EQUATIONS
 - 1.4 EQUATIONS AND INEQUALITIES
 - 1.5 FUNCTIONS AS RULES AND TABLES
 - 1.6 FUNCTIONS AS GRAPHS
 - 1.7 PROBLEM-SOLVING PLAN
 - 1.8 PROBLEM-SOLVING STRATEGIES: MAKE A TABLE; LOOK FOR A PATTERN
-

Overview

Equations and Functions consists of eight lessons that introduce students to the language of algebra.

Suggested Pacing:

Variable Expressions - 1 hr Order of Operations - 1 hr Patterns and Equations - 1 – 2 hrs Equations and Inequalities - 1 – 2 hrs Functions as Rules and Tables - 0.5 hrs Functions as Graphs - 1 hr Problem-Solving Plan - 0.5 hr Problem-Solving Strategies: - 2 hrs *Make a Table; Look for a Pattern*

If you would like access to the Solution Key FlexBook for even-numbered exercises, the Assessment FlexBook and the Assessment Answers FlexBook please contact us at teacher-requests@ck12.org.

Problem-Solving Strand for Mathematics

The problem-solving strategies presented in this chapter, *Make a Table* and *Look for Patterns*, are foundational techniques. *Making a Table* can help structure a student's ability to organize and clarify the data presented and teach the student to communicate more clearly. When teaching *Look for a Pattern*, ask questions such as:

- Can you identify a pattern in the given examples that would let you extend the data?
- Do you observe any pattern that applies to all the given examples?
- Can you find a relationship or operation(s) that would allow you to predict another term?

Alignment with the NCTM Process Standards

Two promising practices, focused on the communication standards, can be effectively used with these strategies. The first is setting aside a time on a regular basis (i.e. once a week) to have students write about the problem solving they have been doing (COM.2). Present a daily warm-up problem which is well suited to the strategy being learned such as *Look for a Pattern*. Students work on one problem a day, first individually and then as a group

with teacher leadership and summation. Each day the problem is discussed and solved, and many different points of view are shared in the process (COM.1, COM.3). At the end of the week, students are asked to write about any one of the problems they did earlier in the week. This practice pushes students to develop logical thinking skills, to benefit from the classroom work that was shared earlier in the week, and to learn to communicate mathematical ideas clearly (COM.4). It also gives students a choice; they only have to write about one of the problems, and it can be the problem that made the most sense to them. Oftentimes, unfortunately, we do not honor enough what makes sense to students; we expect them only to be able to follow the logic presented to them. We must give them experiences of “sense-making” as well (RP.1, RP.2).

A second practice is posting student work, such as:

- gallery walks where work in progress can be viewed
- posters that highlight exceptionally well done solutions
- student essays posted on the classroom wall

What matters is that students’ work is valued and shared (RP.3). If these pieces are large enough to be seen from a distance, this practice helps students to recall various approaches to solving problems (RP.4) and keeps their thinking “alive.”

- COM.1 - Organize and consolidate their mathematical thinking through communication.
- COM.2 - Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
- COM.3 - Analyze and evaluate the mathematical thinking and strategies of others.
- COM.4 - Use the language of mathematics to express mathematical ideas precisely.
- RP.1 - Recognize reasoning and proof as fundamental aspects of mathematics.
- RP.2 - Make and investigate mathematical conjectures.
- RP.3 - Develop and evaluate mathematical arguments and proofs.
- RP.4 - Select and use various types of reasoning and methods of proof.

1.1 Variable Expressions

Learning Objectives

At the end of this lesson, students will be able to:

- Evaluate algebraic expressions.
- Evaluate algebraic expressions with exponents.

Vocabulary

Terms introduced in this lesson:

algebra
generalize
variables
equations
substitution (evaluation, “plugging-in”)
values
solution

Teaching Strategies and Tips

Although the properties of real numbers are not covered until the chapter *Real Numbers*, some basic working assumptions must be made in this chapter. Addition and multiplication involving negative numbers, for example, will probably not be new to the majority of your students, but it is necessary nonetheless for the completion of the Review problems. A comprehensive assessment prior to the end of the first week of classes is suggested so that you can gauge how much time you will need to cover *Equations and Functions* and *Real Numbers*.

Unknowns are used as placeholders in equations like $4x + 3 = x - 2$ and formulas like $A = 2\pi r$. Using unknowns offers great advantages. It prevents having to solve problems “from scratch.” They allow the general formulation of arithmetical laws such as $a + b = b + a$ and $a \times b = b \times a$ for all real numbers a and b . Variables are used as shorthand in functional relationships such as, “Let x represent the number of cricket chirps heard in a 15 minute interval. Then the temperature outside is given by the function $T(x) = mx + b$, where T represents the outside temperature.”

The difference between equations and expressions is important. Basically, one has an equals sign and the other one does not; equations can be solved and expressions are evaluated. It will help the students to see several examples side-by-side:

Equations: $-x + 2 = 7$, $x + 2y = x - 1$, $x = 7$

Expressions: $-x + 2$, $2x + 2y$, πr^2

Coefficients represent repeated addition and exponents represent repeated multiplication.

The process of evaluation is also called *substitution*. It may be explained to the students that since the variable is standing in place of all values (or some unknown value), substitution is an occasion to replace the letter with a value. Concerning notation: It is important for students to understand that parentheses are at their disposal at all times because they make calculations more explicit.

Several examples can be presented, each involving a different aspect of the evaluation process. For example:

- One instance of the variable: Evaluate $-3x + 2$ when $x = -2$
- Two or more instances of the variable: Evaluate $-3x + 7 - x + 1$ when $x = -2$
- Two or more variables: Evaluate $-3x + 7y + 1$ when $x = -2$ and $y = 4$
- Evaluation involving negative numbers and fractions: Evaluate $-\frac{3}{4}x + 7$ when $x = -4$

Examples from geometry and physics allow the students to see the use and application of substitution and even provide motivation in other directions. See lesson Examples 5 and 7.

It is helpful to include words and phrases associated with functions in your lesson. Establishing familiarity with words and phrases like, “input and output,” “...in for x ,” “out of the expression comes...,” “value,” “...the expression evaluated at x ,” etc., will help provide students with a foundation for the upcoming lesson on functions and function notation.

Word or story problems are important to establish right from the beginning.

Error Troubleshooting

Teachers are advised not to allow notation to become an obstacle for their students from the beginning. Writing should be in-line and organized and maintain a logical flow. Students may misread their work if their writing is illegible. $2x + 3$ can look like $2x + 3$, $2x + 3$ like lower-case t , tiny negatives disappear into the paper, other stray marks become minus signs, etc.

Students often compress their answers on paper, especially when they are used to writing across instead of in a downward fashion. $3x - 1 = 8 = 3x = 9 = x = 3$. The equals sign is being used in at least two ways: equality between quantities, and equivalence (or implication) between equations.

When evaluating algebraic expressions, students can easily lose track of negatives.

Evaluate $1 - x$, where $x = -1$.

Evaluate $1 - x^2$, where $x = -1$.

Evaluate $1 - \frac{x-3}{-3}$, where $x = -1$.

The first example involves a variable with a negative “out in front” and the value being substituted is negative. The second involves the vulnerable combination of negatives and exponents. The third involves negatives and fractions. As a check, ask your students if they can tell the difference between

$$-1^2 \quad \text{and} \quad (-1)^2$$

1.2 Order of Operations

Learning Objectives

At the end of this lesson, students will be able to:

- Evaluate algebraic expressions with grouping symbols.
- Evaluate algebraic expressions with fraction bars.
- Evaluate algebraic expressions with a graphing calculator.

Vocabulary

Terms introduced in this lesson:

order of operations

PEMDAS

grouping symbols

parentheses

nested parentheses

Teaching Strategies and Tips

As the lesson illustrates, a string of symbols is meaningless without an order to the operations established beforehand. The usual order of operations – evaluate expressions inside parentheses, exponents, multiplication and division from left to right before addition, and subtraction from left to right – is introduced alongside the options of grouping symbols available to students.

Examples in this lesson illustrate the different kinds of situations that can arise involving grouping symbols: expressions without parentheses; expressions with parentheses (and other grouping symbols); inserting parentheses manually that are otherwise not inherent in the expression; parentheses within parentheses (nested parentheses – grouping symbols several layers deep); working with a complicated-looking expression (working from inner grouping symbol to outer grouping symbol), thereby being convinced that by sticking to the order of operations, its simplification does not need to be difficult.

Students also learn to consider fraction bars as grouping symbols. The fraction bar is an invisible bracket – the numerator and the denominator need to be simplified before proceeding.

Consider evaluating the following with and without adding parentheses:

$$-x^2 + 3, \text{ when } x = [U+0080] [U+0093] 1$$

Teachers are advised to use caution when presenting the mnemonic *PEMDAS* (Parenthesis, Exponent, Multiplication, Division, Addition, Subtraction). Though it is a highly effective way to get students to memorize the order of operations, the horizontal nature of our writing is suggestive of having *M* performed before *D*, and *A* before *S*. As you know, it's not multiplication before division or addition before subtraction. Students have responded positively to the vertical schematic:

P
E
M-D
A-S

Error Troubleshooting

As discussed previously, there is a potential for error in the horizontally written mnemonic, *PEMDAS*. Students, especially those who have consciously or unconsciously shut out mathematics and refuse to participate in it in any meaningful way, will inevitably ignore the left-to-right precept and go with what is suggested: *M*. before *D*, and *A* before *S*.

Students should be reminded occasionally that subtracting negatives is equivalent to adding positives.

1.3 Patterns and Equations

Learning Objectives

At the end of this lesson, students will be able to:

- Write an equation.
- Use a verbal model to write an equation.
- Solve problems using equations.

Vocabulary

Terms introduced in this lesson:

horizontal axis

vertical axis

increases

decreases

Teaching Strategies and Tips

The process of taking givens and translating them into mathematical statements by way of symbols is called modeling. Use the four-step problem-solving process to break down problems.

Step 1: Extract the important information.

Step 2: Translate into a mathematical equation.

Step 3: Solve the equation.

Step 4: Check the result.

When it comes to providing a list of verbal cues and their mathematical translation, it is helpful to complement the lesson with a more comprehensive list:

at least

at most

as much as

no more than x

increased by y

x decreased by y

sum of x and y

total of x and y

x plus y

gain

raise

more

increase of

difference between x and y

x minus y

loss

less

fewer

take away

multiply x by y

product of x and y

x times y

double x

twice x

triple x

percent of x

fraction of x

quotient of x and y

divide x by y

x divided by y

per

Part of the learning process consists of learning from mistakes. If students are told that their answers are wrong without good reasons or explanations, they will simply lose out on what could have been a learning experience. Consider the following subtle example.

Students were asked to simplify the expression:

$$(-x)(-x)(-x)$$

One student gave the answer:

$$(-x)^3$$

The exponent is odd, $(-x)^3$ so it can be further simplified to $-x^3$. The two expressions can be interpreted differently if order of operations is the focus:

$(-x)^3$ means to cube the opposite of x

$-x^3$ means to take the opposite of x cubed

1.3. PATTERNS AND EQUATIONS

Error Troubleshooting

Require students to check their work. Some beneficial questions are: *Does the solution make sense? How does it compare with a quick estimate I could make?*

Some students will forget to follow the order of operations. Encourage students to read the problem carefully. Example 3 from this lesson demonstrates how to use a pattern as a tool to develop the required equation.

1.4 Equations and Inequalities

Learning Objectives

At the end of this lesson, students will be able to:

- Write equations and inequalities.
- Check solutions to equations.
- Check solutions to inequalities.
- Solve real-world problems using an equation.

Vocabulary

Terms introduced in this lesson:

inequality

constants

greater than

less than

greater than or equal to

less than or equal to

not equal to

Teaching Strategies and Tips

This lesson can be viewed as a generalization of the previous lesson. The emphasis here is placed on defining the variable correctly, then translating the words in the problem into a mathematical expression that contains those variables, using an equal sign or an inequality.

Checking solutions to inequalities is inherently the same as checking solutions to equations. The variable is similarly replaced by a value and should produce a true statement.

The process of simplifying equations can be stated as follows:

- a. Evaluate any parentheses, exponents, multiplications, divisions, additions, and subtractions according to the usual order of operations. Make proper use of the associative and distributive laws.
- b. Combine *like terms*. This means adding or subtracting variables of the same type.
- c. Add, subtract, multiply by, or divide by any value (except 0) on both sides of the equation.

The exception, of course, is when multiplying or dividing an inequality by a negative, reversing the inequality symbol in order to obtain an equivalent inequality. It is useful to present examples to students in which the inequality symbol does not get reversed and others in which the symbol must be reversed.

Error Troubleshooting

Have students write out and identify the givens in problems.

If variables other than x are used students need to be careful that Z [U+0080] [U+0099]s and 2 [U+0080] [U+0099]s are not confused.

Students have trouble labeling the variables correctly. Have them locate the key question in the exercise such as: *What? How much? When? How long? Where? How far?*

Students should be cautioned to avoid the common mistake when fractions are involved; failing to divide the entire numerator will lead to incorrect results. For example:

$$\frac{2x+10}{2} \neq x + 10$$

1.5 Functions as Rules and Tables

Learning Objectives

At the end of this lesson, students will be able to:

- Identify the domain and range of a function.
- Make a table for a function.
- Write a function rule.
- Represent a real-world situation with a function.

Vocabulary

Terms introduced in this lesson:

function
independent variable
dependent variable
input
output
domain
range
table

Teaching Strategies and Tips

This lesson introduces students to the concept of a function. Functions are relationships, associations, dependences, between two quantities. Independent and dependent variables are distinguished. Students realize that not all real numbers are permissible in the functions they write for problems. They learn to identify the domain and range of a function. Consider the level of detail in the explanation to Example 1 from the SE.

Tables provide students with organizational tools for dealing with functions. Students are asked to make tables for explicit functions. Conversely, students are asked to write a function rule or expression from a given table of values.

Emphasis is placed on representing real-world situations with functions. Examples include, the price a person pays for phone service depending on the number of minutes used, and the number of amusement park rides someone goes on depending on the total amount of money paid.

Note: teachers frequently must explain that there is no implied multiplication in the notation: $f(x)$, read “ f of x ”.

Some examples include:

- The cost of postage for a letter is a function of its weight.

- The amount of flour used when preparing a recipe is a function of the number of servings desired.
- How much you pay for gas at a station is a function of the number of gallons pumped.

Emphasize representing real-world situations with functions by writing out the function in words first. For example,

$$\text{total cost} = \text{flat fee} + \text{hourly fee} \times \text{number of hours}$$

Error Troubleshooting

Function notation can be an obstacle for many students, primarily because they do not understand its purpose. Some students will try to multiply the f and the x .

1.6 Functions as Graphs

Learning Objectives

At the end of this lesson, students will be able to:

- Graph a function from a rule or table.
- Write a function rule from a graph.
- Analyze the graph of a real world situation.
- Determine whether a relation is a function.

Vocabulary

Terms introduced in this lesson:

coordinate plane
Cartesian plane
origin

coordinate points
quadrants
vertical line test

Teaching Strategies and Tips

This lesson introduces students to the coordinate plane and graphing a function from a rule or table.

Example 1: Some students will graph the ordered pairs incorrectly. Emphasize starting at the origin, moving left or right, moving up or down, and then plotting the given point.

It is a rule that functions specify the relationship between the independent and the dependent variables in a problem. Students usually understand that they must use the value of the independent variable to get the values of the dependent variable, but they should also understand that functions can be expressed verbally (in words), symbolically (as an equation), numerically (as a table of values), and graphically (as a graph).

When students are determining whether a relation is a function, they are essentially noticing that for each input there is exactly one output. This can take many forms, depending on how the function is presented: Two or more numbers in the dependent column of a table; a graph intersecting the same vertical line in at least two places; etc. Students ought to be able to state the definition of a function in their own words, and teachers perhaps ought to be able to do this in at least a few different ways. For example, each x -value has only one y -value assigned to it, a value in the

range can belong to more than one element in the domain, etc. Being a function would mean that more than one person in a class can have the same height, but one person cannot have two or more heights.

As stated in the lesson, it is wise to work with a lot of values when you start graphing, but as students learn about different types of functions they will find that they will only need a few points in the table of values. This will help prevent substitution and arithmetical errors. If the points from a table do not lie on a line (assuming linear functions), then the student knows that one of the calculations has an error in it. Plotting too few points won't catch these kinds of mistakes.

Furthermore, a table of values should include inputs from both sides of the number line – negatives, positives, and zero. Sometimes it is evident that negative numbers cannot be used for the independent variable because of domain considerations.

Error Troubleshooting

Example 1: Have students place arrow heads (\leftrightarrow , \updownarrow) above the points in an ordered pair to help them remember which direction to move when plotting the point.

Functions cannot have more than one output per input, but two inputs can have the same input. Examples that clarify this will help the student understand the definition of a function.

1.7 Problem-Solving Plan

Learning Objectives

At the end of this lesson, students will be able to:

- Read and understand given problem situations.
- Make a plan to solve the problem.
- Solve the problem and check the results.
- Compare alternative approaches to solving the problem.
- Solve real-world problems using a plan.

Vocabulary

Terms introduced in this lesson:

“read and understand the problem”

knowns

unknowns

“solve the problem”

“check and interpret the answer”

Teaching Strategies and Tips

In this lesson, students are introduced to problem solving and a new plan that will be used throughout the book.

Some common strategies that students will learn are listed here for easy reference:

- *draw a diagram*
- *make a table*
- *look for patterns*
- *guess and check*
- *work backward*
- *use a formula*
- *read and make graphs*
- *write equations*
- *use linear models*
- *use dimensional analysis*
- *use the right type of function for the situation*

It is helpful to advise students that when translating sentences into expressions and equations, they should not attempt to solve the entire problem all at once. Instead, students should read the problem “phrase by phrase”; translate one phrase, then move on to the next.

Teachers can ask their students to reread the question, rephrase it, or even read it aloud.

Estimation is a good habit to develop as it helps confirm the answers students obtain after solving a problem.

Error Troubleshooting

The following list includes possible sources of mistakes for students while solving an applied problem:

- *Not listing the given information*
- *Not restating the question being asked in your own words*
- *Not selecting a variable to represent the unknown quantity*
- *Not clearly stating what the variable represents*
- *Not looking for possible patterns*
- *Not looking up a definition or formula*

A common mistake students make is checking their answers in the equations that they constructed instead of in the original wording of the problems.

1.8 Problem-Solving Strategies: Make a Table; Look for a Pattern

Learning Objectives

At the end of this lesson, students will be able to:

- Read and understand given problem situations.
- Develop and use the strategy: *Make a Table*.
- Develop and use the strategy: *Look for a Pattern*.
- Plan and compare alternative approaches to solving the problem.
- Solve real-world problems using selected strategies as part of a plan.

Vocabulary

Terms introduced in this lesson:

“make a table”

“look for a pattern”

Teaching Strategies and Tips

In this lesson students learn how to develop and use the methods: *make a table* and *look for a pattern*.

Teaching Tip: Review key phrases with students such as: *What? How much? When? How long? Where? How far?* (and even *Who?* and *Why?*). If students’ answers (with appropriate units) do not answer the key phrase in the problem, then the student needs to return to it and either back-substitute to obtain the correct answer or start over by labeling the variables differently to answer the key phrase.

In Example 1, the key phrase is *How much was the total bill?* By distilling this question from the rest of the problem, a student understands what is unknown: *the total bill* (how much the pizza costs). At this point, a variable can clearly be defined.

The strategy *make a table* enables the problem-solver to recognize patterns and relationships within numerical data organized in tables. Students sometimes need help with table headings. In Example 2, the units are *weeks*, *minutes per day*, and *minutes per week*. By extracting this information and laying it out on paper, a student can readily infer the column headings.

Look for a pattern has the problem-solver create only a minimum of rows until a pattern can be established and used to obtain the answer.

Error Troubleshooting

The following list includes possible sources of mistakes for students while solving an applied problem:

- *Not listing the given information*
- *Not restating the question being asked in your own words*
- *Not selecting a variable to represent the unknown quantity*
- *Not clearly stating what the variable represents*
- *Not looking for possible patterns*
- *Not looking up a definition or formula*

CHAPTER 2

TE Real Numbers

CHAPTER OUTLINE

2.1 INTEGERS AND RATIONAL NUMBERS

2.2 ADDITION OF RATIONAL NUMBERS

2.3 SUBTRACTION OF RATIONAL NUMBERS

2.4 MULTIPLICATION OF RATIONAL NUMBERS

2.5 THE DISTRIBUTIVE PROPERTY

2.6 DIVISION OF RATIONAL NUMBERS

2.7 SQUARE ROOTS AND REAL NUMBERS

2.8 PROBLEM-SOLVING STRATEGIES: GUESS AND CHECK; WORK BACKWARD

Overview

In this chapter, students solve real-world problems involving addition, subtraction, multiplication, and division of real numbers; make use of the number line, absolute value, and square roots; and solve equations through the application of additive inverses and reciprocals. In the final lesson, students learn the methods of *guess and check* and *working backwards*.

Suggested Pacing:

Integers and Rational Numbers - 1 hr Addition of Rational Numbers - 1 hr Subtraction of Rational Numbers - 1 hr Multiplication of Rational Numbers - 1 hr The Distributive Property - 1 hr Division of Rational Numbers - 1 hr Square Roots and Real Numbers - 1 hr Problem-Solving Strategies: *Guess and Check; Work Backward* - 1 hr

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Problem-Solving Strand for Mathematics

The problem-solving strategies presented in this unit, *Guess and Check* and *Work Backwards*, invite students to use their mathematical intuition and common sense in making sense of everyday experiences with mathematics.

The technique of *Guess and Check* formalizes what many students have been doing for years: making intelligent guesses and, often, coming up with the right answer. The technique *Working Backwards* lays the groundwork for solving multi-step equations and helps students understand the need for using opposite operations.

Aligning with the NCTM Process Standards

In the NCTM process standards, the concepts of Connections and Representation receive special attention. In *Guess and Check* and *Working Backwards*, students recognize and use connections among mathematical ideas (CON.1), especially when encouraged to look for and think about patterns as they work through the processes. As they write

equations that express the reality of the problem situation (R.2 and R.3), they use the language of mathematics to express mathematical ideas precisely (COM.4).

- COM.4 - Use the language of mathematics to express mathematical ideas precisely.
- CON.1 - Recognize and use connections among mathematical ideas.
- PS.3 - Apply and adapt a variety of appropriate strategies to solve problems.
- RP.2 - Make and investigate mathematical conjectures.
- RP.4 - Select and use various types of reasoning and methods of proof.
- R.2 - Select, apply, and translate among mathematical representations to solve problems.
- R.3 - Use representations to model and interpret physical, social, and mathematical phenomena.

2.1 Integers and Rational Numbers

Learning Objectives

At the end of this lesson, students will be able to:

- Graph and compare integers.
- Classify and order rational numbers.
- Find opposites of numbers.
- Find absolute values.
- Compare fractions to determine which is bigger.

Vocabulary

Terms introduced in this lesson:

integers

whole numbers

greatest

least

even/odd numbers

rational numbers

ratio

numerator

denominator

proper fractions

improper fractions

equivalent fractions

reducing

simplest form

lowest common denominator

common denominator

opposites

absolute value

entire expression

Teaching Strategies and Tips

Through the use of the number line in Example 1, students learn that “greater than” and “farther to the right” are equivalent, and that “less than” and “farther to the left” are equivalent.

Use Example 2 to show that odd numbers are two apart. The frog will never land on an even number because:

- the frog begins at an odd number
- the frog only makes a jump of 2

Students get a visual reinforcement that odd numbers are always two hops apart, *even though the number 2 is even*. This will help them understand later why two consecutive, *odd* numbers can be written as x and $x + 2$.

In Example 4, ask students to divide the pie diagrams into *equal* parts according to the denominators of each fraction and to *shade* according to their numerators. Common denominators have not been introduced at this point.

Use Example 5 to contrast Example 4 and motivate the process of finding common denominators. The problem of determining the larger of two, nearly equal fractions is a harder one. This task becomes straightforward once students learn to rewrite fractions as equivalent ones. Note the switch from “pies” in Example 4 to “pie rectangles” in Example 5.

Determining the larger of two rational numbers may be expressed symbolically: Given $\frac{a}{b}$ and $\frac{c}{d}$, each is equivalent to $\frac{ad}{bd}$ and $\frac{cb}{bd}$ respectively, where the choice of common denominator will never need to be larger than bd . Therefore, if $ad > cb$, then the fraction $\frac{a}{b}$ is larger. If $ad < cb$, then the fraction $\frac{c}{d}$ is larger.

Additional examples:

- Which is greater $\frac{9}{13}$ or $\frac{2}{3}$?

Solution: Since $9 \cdot 3 = 27 > 26 = 13 \cdot 2$, then $\frac{9}{13} > \frac{2}{3}$.

- Which is greater $\frac{31}{7}$ or $\frac{40}{9}$?

Hint: Convert to mixed-numbers and see Example 5.

- Challenge: Without a calculator, arrange from least to greatest. $\frac{7}{19}, \frac{5}{17}, \frac{1}{3}$.

Hint: Compare any two fractions. Compare the larger of these two with the third.

A mirror placed at the origin, perpendicular to the number line, reflects each whole number the same distance into the mirror as the distance it measures from the mirror. This demonstration shows that every real number has an opposite, *with the exception of zero*. One of the more common questions teachers get from their students is whether zero is positive or negative; it is clear from the demonstration above that zero has no reflection and therefore zero cannot be either. Positives and negatives are reflections of each other. See Example 7.

Distance is briefly explained in terms of absolute value. They are covered in the chapter *Graphing Linear Inequalities*.

Use Example 8 to show that absolute value expressions are grouping symbols and the order of operations applies when evaluating them – students must treat them like a parenthesis in that what’s inside must be simplified first.

Error Troubleshooting

General Tip: Often, when students cancel *all* the factors in a numerator, they write an answer of 0, since nothing is left. For example:

$$\frac{6}{30} = \frac{\cancel{2} \cdot \cancel{3}}{\cancel{2} \cdot \cancel{3} \cdot 5} \neq \frac{0}{5}$$

When canceling repeated factors from the numerator and denominator of a fraction, remind students that a 1 remains.

In Example 7e, urge students to apply the -1 to the *whole* expression since “opposite of” means multiplying an entire expression by -1 .

Example:

- The opposite of $-2x + 1$ is $2x - 1$ and not $2x + 1$.

General Tip: Parentheses are an organizational tool. Students are encouraged to put them around an expression if it helps them prevent the above mistake.

Example:

- Find the opposite of $-2x + 1$.

Solution:

Start by putting parentheses around the expression: $-(-2x + 1)$

Use Example 8d to help students see that $-|-15| \neq 15$ and in general, $-|x| \neq |x|$. Overly confident students reciting that *the absolute value is always positive* can incorrectly simplify $-|x|$ as $|x|$ instead of $|-x|$ as $|x|$.

2.2 Addition of Rational Numbers

Learning Objectives

At the end of this lesson, students will be able to:

- Add using a number line.
- Add rational numbers.
- Identify and apply properties of addition.
- Solve real-world problems using addition of fractions.

Vocabulary

Terms introduced in this lesson:

adding a negative number

LCD/LCM

least common multiple

equivalent fractions

commutative property

associative property

additive identity

additive properties

Teaching Strategies and Tips

Use Examples 1-3 to provide a visual frame of reference for adding real numbers. Eventually, students will need to add numbers without using the number line.

General Tip: Move to the left on the number line when faced with a negative number or subtraction.

Additional example:

- *Represent the sum $-7 - 5$ on the number line.*

Solution:

Starting at -7 , move 5 units to the left: $-7 - 5 = -12$

Use Examples 4 and 5 to introduce LCD (lowest common denominator) and LCM (lowest common multiple). Although they have different meanings, the difference is subtle.

Additional examples:

- Find the LCM of 24 and 30.

Solution:

The lowest number that both 24 and 30 divide into without remainder is 120.

- Simplify $\frac{5}{24} + \frac{1}{30}$.

Solution:

In order to combine these fractions we need to rewrite them over a common denominator. We are looking for the lowest common denominator (LCD). We need to first identify the lowest common multiple (LCM) of 24 and 30.

$$\frac{5}{24} + \frac{1}{30} = \frac{25}{120} + \frac{4}{120} = \frac{29}{120}$$

Teachers are encouraged to reinforce the notions of LCD and LCM using visual representations as illustrated in Example 4.

To help students learn the difference between the associative and commutative properties state the rule that is used when doing each example in the classroom and then have them note the differences.

Error Troubleshooting

General Tip: Students who have difficulty with LCD and LCM at this point, might benefit from *factor trees*, which are discussed in the next lesson.

For instance, in Example 6, students are asked to combine two denominators having a common factor. The factor trees for 12 and 9 are:

- $12 = 2 \times 2 \times 3$
- $9 = 3 \times 3$

To find the LCM, start by taking all of the prime numbers $\{2, 2, 3\}$ of 12. Then ask what prime numbers of 9 are missing from this list? The second 3 is missing. Therefore, the LCM of 12 and 9 is the product of the numbers in the list $\{2, 2, 3, 3\}$

$$2 \times 2 \times 3 \times 3 = 36$$

General Tip: Students need constant reminding that adding fractions cannot take place without common denominators. Although memorizing rules such as,

- $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$
- $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

may prove helpful, teachers are encouraged to explain *why* denominators must be the same. Use an approach similar to the following:

- Measurements involving feet and inches cannot be added until a common unit is chosen. For example, the sum of 7 inches and 3 feet is not equal to 10 inches, 10 feet, or 10 feet – inches. Converting 3 feet to 36 inches allows us to add the 7 inches to it: $7 \text{ in} + 36 \text{ in} = 43 \text{ in}$.

2.2. ADDITION OF RATIONAL NUMBERS

- The same is true with fractions. “Halves” and “fourths” must be first converted to the common unit “fourths.” Similarly, fractions representing “halves” and “thirds,” will need to be converted to a third unit, “sixths,” the LCM of 2 and 3.
- Example 4 uses a visual argument to justify the need for common denominators. In one direction, the “cake” is sliced according to the first fraction, $3/5$; in the other direction it is sliced according to the second fraction $1/6$. The sum of $3/5$ and $1/6$ can now be found – but only in terms of the little squares since the total number of squares represent the common unit.

2.3 Subtraction of Rational Numbers

Learning Objectives

At the end of this lesson, students will be able to:

- Find additive inverses.
- Subtract rational numbers.
- Evaluate change using a variable expression.
- Solve real world problems using fractions.

Vocabulary

Terms introduced in this lesson:

additive inverse

opposite

opposite/inverse process

subtraction

method of prime factors

simplify

simplest form

invisible denominator

speed

difference

change

Teaching Strategies and Tips

General Tip: Additive inverses provide a way to subtract two real numbers. Students learn that the operation of subtraction can be replaced with addition by *adding the additive inverse of the number*.

Subtraction by adding additive inverses applies as well to fractions.

Example:

$$\frac{3}{7} - \frac{6}{7} = \frac{3}{7} + \left(-\frac{6}{7}\right) = \frac{3+(-6)}{7} = \frac{-3}{7}$$

Although it may be easier for students to keep it as subtraction:

$$\frac{3}{7} - \frac{6}{7} = \frac{3-6}{7} = \frac{3+(-6)}{7} = \frac{-3}{7}$$

Tips on factor trees:

- Do not include the trivial factors of 1 in the branches of factor trees.
- The first two branches of the factor tree can be any pair of factors.
- For instance, in Example 2, $90 = 9 \times 10$ and $126 = 9 \times 14$. Students may choose differently: $90 = 45 \times 2$ and $126 = 21 \times 6$.
- To find the LCM of three numbers, no more than three factor trees need to be constructed. In general, there will be as many factor trees as numbers whose LCM is being sought. See Example 4 for three numbers.

In Example 4, the “invisible denominator” is a useful way of getting students to realize that

$$x = \frac{x}{1}$$

That is, integers are rational numbers. Every whole number can be viewed as a rational number whose denominator is 1.

Use Examples 5 and 6 to demonstrate the concept of change.

- A variable expression is used to evaluate *change in speed* and *change in light intensity*, respectively. The functions are provided already; and therefore do not need to be derived from scratch.
- The focus is on the concept of change, which will be new for many students. Exploration is encouraged through tables and graphs.
- To find the change in a quantity using a function, cast this as a subtraction problem: *subtract first quantity from second quantity*.
- Explaining the results in words will benefit students. *Positive change* in speed means an increase in speed. The change is *negative* in Example 6: the intensity *reduces* as the train travels farther away from the light source.

Error Troubleshooting

General Tip: For many students, the combination of subtraction and finding common denominators is prone to errors. Practice is crucial. See Examples 3 and 4 as well as *Review Questions* 1a – 1i.

Example 5 and 6: Change = second quantity – first quantity; i.e., subtract the first quantity *from* the second quantity. In context:

- Change = Speed 2 – Speed 1
- Change = Intensity 2 – Intensity 1

2.4 Multiplication of Rational Numbers

Learning Objectives

At the end of this lesson, students will be able to:

- Multiply by -1 .
- Multiply rational numbers.
- Identify and apply properties of multiplication.
- Solve real-world problems using multiplication.

Vocabulary

Terms introduced in this lesson:

argument

multiplicative properties

multiplicative identity property

distributive property

Teaching Strategies and Tips

Changing the sign of a number is equivalent to multiplying it by -1 . In Example 1, students find the opposite of several numbers and expressions being careful to use parentheses when appropriate and simplifying the result.

Use Examples 2 and 3 to justify *why*, when multiplying two fractions, the numerators multiply together and the denominators multiply together.

Use the product of *three or more* fractions as in Examples 4c and 4d as an extension of the multiplication rule.

Additional examples:

- *Multiply the following rational numbers.*

$$\frac{3}{11} \cdot \frac{5}{7}$$

Hint: *The product of two rational numbers is the product of their numerators divided by the product of their denominators.*

- *Multiply the following rational numbers.*

$$\frac{11}{5} \cdot \frac{7}{4} \cdot \frac{3}{10}$$

Hint: Multiply all the numerators and all the denominators. Do not convert the improper fraction to mixed form.

- *Multiply the following rational numbers.*

$$\frac{5}{7} \cdot 12$$

Hint: Rewrite the 12 as $12/1$, using the “invisible 1”.

Students first learn about the convenience of canceling *before* multiplying in Examples 4d and 5.

Additional example:

- *Multiply the following rational numbers.*

$$\frac{24}{33} \cdot \frac{8}{27} \cdot \frac{9}{64}$$

Solution:

$$\frac{24}{33} \cdot \frac{8}{27} \cdot \frac{9}{64} = \frac{3 \cdot 8}{3 \cdot 11} \cdot \frac{8}{3 \cdot 9} \cdot \frac{9}{8 \cdot 8} = \frac{\cancel{3} \cdot \cancel{8}}{\cancel{3} \cdot 11} \cdot \frac{\cancel{8}}{3 \cdot \cancel{9}} \cdot \frac{\cancel{9}}{\cancel{8} \cdot \cancel{8}} = \frac{1}{33}$$

Use Examples 6-8 to introduce the four properties of real numbers which involve multiplication: the commutative, associative, multiplicative identity, and distributive properties.

- A geometric interpretation of the commutative property is to consider finding the area of a rectangle. $L \times W$ is the same number no matter how you draw the rectangle or what you call L and W ; therefore, $L \times W = W \times L$. Similarly, the commutative property says that the order for multiplying any two real numbers does not matter. See Example 6.
- The associative property of multiplication concerns three or more numbers. Just as for addition, the sum is the same regardless of *how they are grouped* and in which pair the multiplication takes place first.
- State the rule being used in each example you do in the classroom.

Error Troubleshooting

Example 1b: The opposite of π is simply $-1 \cdot (\pi) = -\pi$. There is no need to use the decimal expansion.

Example 1c: Multiply both terms of the expression by -1 . This will make more sense to students after covering the distributive law in the next lesson.

Additional example:

- *Find the opposite of the expression*

$$x - 4y + 1$$

Hint: multiply each of the three terms by -1 .

The difference between absolute value and other grouping symbols is that multiplying absolute value by -1 will not affect the argument; that is, a negative will not distribute into the absolute value. See Example 1d.

General Tip: It is helpful to note that

- $|x|$ and $|-x|$ are always positive
- $-|x|$ is always negative.

General Tip: A common mistake is to forget to cancel like factors *before* multiplying the fractions, as the numbers will only get larger and thus harder to factor. Have students factor numerators and denominators first to remove any repetitions by canceling. Then carry out the remaining easier multiplication.

2.5 The Distributive Property

Learning Objectives

At the end of this lesson, students will be able to:

- Apply the distributive property.
- Identify parts of an expression.
- Solve real-world problems using the distributive property.

Vocabulary

Terms introduced in this lesson:

distributive property

Teaching Strategies and Tips

In the lesson introduction:

- Students see that distributing gift bags among the children is equivalent to *distributing numbers*.
- Care must be taken in using the abbreviations p , f , and c for *photo*, *favor*, and *candy*, respectively. These are not variables but units of measurement. Students can misconstrue the notation and intention – these quantities are not unknown and do not need to be solved for.

Use Examples 1 and 2 to show that some expressions can be simplified in more than one way. Use the distributive property and apply the order of operations to obtain the same result.

Additional examples:

- *Use the distributive property to simplify:*

$$-4(5 - 7)$$

Solution: $-4(5 - 7) = -4 \cdot 5 + (-4) \cdot (-7) = -20 + 28 = 8$

Note: Use care with the negatives.

- *Use order of operations to simplify:*

$$-4(5 - 7)$$

Solution: $-4(5 - 7) = -4(-2) = 8$

In Examples 2 and 3b-3d, have students deal with the minus sign by committing to (1) using additive inverses, or (2) subtraction, but not both. For instance, Example 3b can be solved in two ways:

- $7(3x - 5) = 7(3x + (-5)) = 7(3x) + 7(-5) = 21x - 35$
- $7(3x - 5) = 7(3x) - 7 \cdot 5 = 21x - 35$

Use Example 4 to demonstrate the *hidden distributive property*. Fractions with two or more terms in their numerators can be simplified using the distributive property. The hidden distributive property is based on the rule for multiplication of rational numbers:

$$\frac{a}{b} = \frac{1}{b} \cdot a$$

Point out in Example 5 that the distributive property equally applies on the right. *Steel required* = $(4 + 5)8$ is also a correct answer.

Additional example:

- *Simplify using the distributive property.*

$$(3x - 1)2$$

Solution: $(3x - 1)2 = (3x)2 - 1 \cdot 2 = 6x - 2$

In Example 6,

- Some students will want to round up rather than down. Have students complete the problem by rounding up to see that they would not have enough money.
- Point out that the distributive property equally applies to factors with three or more terms.

Error Troubleshooting

In Example 3d, the x 's will eventually cancel. Students can forget to reduce in the end.

The 3 in Example 4b does not cancel. This is a common mistake made by students.

Additional example:

- *Simplify the following expression.*

$$\frac{ax - 2}{a} \neq \frac{\cancel{a}x - 2}{\cancel{a}} = x - 2$$

Solution: $\frac{ax-2}{a}$ can be rewritten as $\frac{1}{a}(ax-2)$ Distribute the $\frac{1}{a}$:

$$\frac{1}{a}(ax-2) = \frac{ax}{a} - \frac{2}{a} = x - \frac{2}{a}$$

2.6 Division of Rational Numbers

Learning Objectives

At the end of this lesson, students will be able to:

- Find multiplicative inverses.
- Divide rational numbers.
- Solve real-world problems using division.

Vocabulary

Terms introduced in this lesson:

multiplicative inverse

reciprocals

invert the fraction

improper fraction

invisible denominator

speed

distance

time

Teaching Strategies and Tips

Draw an analogy between the division and subtraction of rational numbers.

- A subtraction problem can be recast as an addition problem using additive inverses (opposites). A division problem can be recast as a multiplication problem using multiplicative inverses (reciprocals).
- When a number is added to its opposite, the additive identity, 0, is obtained. When a number is multiplied by its reciprocal, the multiplicative identity, 1, is obtained.

In Example 1c, remind students that a mixed number needs to be converted to an improper fraction before determining the multiplicative inverse.

Error Troubleshooting

In Example 1d, point out that finding the multiplicative inverse of the expression will not affect the negative. See also Example 2d.

- The reciprocal of $-\frac{x}{y}$ is $-\frac{y}{x}$. (Invert the fraction.)
- The opposite of $-\frac{x}{y}$ is $\frac{x}{y}$. (Multiply by -1 .)

2.7 Square Roots and Real Numbers

Learning Objectives

At the end of this lesson, students will be able to:

- Find square roots.
- Approximate square roots.
- Identify irrational numbers.
- Classify real numbers.
- Graph and order real numbers.

Vocabulary

Terms introduced in this lesson:

square root
principal square root
positive square root
radicals
perfect squares
prime factors
rational number
irrational number
approximate answer
approximation
non-repeating decimals
simplest form
integer
sub-intervals

Teaching Strategies and Tips

Contrary to what students think, there is no discrepancy between there being two possible values for b , given a positive x , such that $b^2 = x$ and only one value for \sqrt{x} . The positive number b is the *principal* square root. It is the value of the function.

Point out that $\sqrt{\quad}$ and $\sqrt[2]{\quad}$ mean the same thing. The $\sqrt[2]{\quad}$ is understood.

In Example 1:

- Use factor trees to break down the radicands into as many perfect squares as possible.

- Primes which appear an even number of times constitute a perfect square.
- Any unpaired factors are left under the radical sign.
- The convention is to leave any irreducible radical in the form: (square root) $\sqrt{\text{(irreducible part)}}$

Lots of practice early on makes it easier for students when the radicals become more complicated and involve variable expressions.

Use Example 3 to multiply, divide, and simplify radical expressions. State which rules were used in classroom examples.

In Example 4, students use a calculator and round their answer to three decimal places. Review rounding decimals.

Motivate irrational numbers in Example 5.

- Irrational numbers complete the set of real numbers.
- They cannot be expressed as ratio of two integers.
- They have an unending (non-terminating), seemingly random (non-repeating) decimal expansion.
- Some irrational numbers: π , $\frac{1}{\pi}$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{\text{any prime}}$

In Example 5, students identify the given numbers. A number is rational if it

- can be expressed as a fraction of integers.
- has a finite decimal expansion.
- has a repeating block of digits in its decimal expansion.

Have students check their calculator displays for *repeating blocks of digits*, not just patterns. For example, the two numbers below have obvious patterns, but no repeating blocks, and therefore are irrational:

- 0.010010001000010000010000001...
- 0.12345678910111213141516171819202122...

Remind students that integers can be written as fractions with a 1 in the denominator. See Example 6a and 6b. The strategy in Example 6e is to simplify first.

The strategy in Example 8 is to use a calculator to find the decimal expansions of the numbers rounded to as many places as is needed to identify each. Since all the numbers are between 3.1 and 3.2, going out to three decimal places is sufficient.

Error Troubleshooting

In Problem 3 of the *Review Questions*:

- Students should find repeating blocks of digits before claiming that a number is rational.
- It is not sufficient to claim that the “unpredictable” decimals of a number on a calculator display belong to an irrational number. The decimal expansion of $2/19$ has 18 seemingly random digits in its repeating block; therefore, the rational number $2/19$ would go “unchecked” on an ordinary calculator display because it could not show enough digits.

2.8 Problem-Solving Strategies: Guess and Check; Work Backward

Learning Objectives

At the end of this lesson, students will be able to:

- Read and understand given problem situations.
- Develop and use the strategy: *Guess and Check*.
- Develop and use the strategy: *Work Backward*.
- Plan and compare alternative approaches to solving problems.
- Solve real-world problems using selected strategies as part of a plan.

Vocabulary

Terms introduced in this lesson:

guess and check
working backwards

Teaching Strategies and Tips

Use Example 1 to introduce *guess and check*. Teachers are encouraged to postpone solutions involving systems of equations (two variables) until chapter *Solving Systems of Equations and Inequalities*.

Allow students to strategize from their guesses. The guessing process will often lead to unexpected patterns that can serve to make better guesses along the way:

- In Example 2, one guess yields a sum of 24, which is half of the desired 48; therefore, the initial numbers should be multiplied by 2.
- In Example 4, note the pattern given by the relation: when Nadia's age is decreased by 1, her father's age decreases by 4. This observation leads to the answer.
- In Example 6, allow students to keep guessing until the *total costs* are the same. Students may notice however that for an increase of 10, the difference between total costs falls by \$1.

Use Example 3 to show how to *work backward*. Reverse the steps starting with the result until the unknown is obtained.

General Tip: Teachers are encouraged to compare alternative approaches to some of the problems.

Error Troubleshooting

General Tip: Remind students to check their work. Ask: *Does the answer make sense?*

CHAPTER 3

TE Equations of Lines

CHAPTER OUTLINE

3.1 ONE-STEP EQUATIONS

3.2 TWO-STEP EQUATIONS

3.3 MULTI-STEP EQUATIONS

3.4 EQUATIONS WITH VARIABLES ON BOTH SIDES

3.5 RATIOS AND PROPORTIONS

3.6 SCALE AND INDIRECT MEASUREMENT

3.7 PERCENT PROBLEMS

3.8 PROBLEM-SOLVING STRATEGIES: USE A FORMULA

Overview

Rules for solving one-step, two-step, and multi-step equations, and equations with variables on both sides are presented. By the end of the fourth lesson, students solve applied problems involving general linear equations. Students reason with ratios and percents, construct proportions and apply them to scaled drawings, and use formulas as a problem-solving strategy.

Suggested Pacing:

One-Step Equations - 1 hr

Two-Step Equations - 1 hr

Multi-Step Equations - 1 hr

Equations with Variables on Both Sides - 1 hr

Ratios and Proportions - 1 hr

Scale and Indirect Measurement - 1 hr

Percent Problems - 1 hr

Problem-Solving Strategies: *Use a Formula* - 1 hr

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Problem-Solving Strand for Mathematics

Use a Formula is the problem-solving strategy highlighted in this unit. Many teachers and students believe that knowing what formula to use or apply to a given problem turns what may have been a difficult “problem” into more of a straightforward exercise. Substituting in known quantities and using a formula, they reason, is simple.

For some students, however, following the steps of a formula is not a simple process, especially if there’s an unknown quantity in the formula. Still for others, using a formula with unfamiliar terms such as ohms or finding the principal

using a formula such as $I = Prt$, is simply baffling. The first case, of course, is an ideal time to encourage students to apply a previously practiced skill like *Working Backwards*. By using opposite operations to “undo” an equation, students acquire a new formula, one they have generated themselves.

When presenting formulas to students, an excellent practice is to help students understand where the formula comes from in the first place. A straightforward example of this would be the formula $P = 2l + 2w$. Students can easily understand that this formula for the perimeter of a rectangle matches the physical process of “marching around” the figure in fact or in their mind. Similarly, a mental image of a grid on which a rectangle is outlined verifies why the area formula for a rectangle is $A = lw$.

The challenge for us as teachers is to relate as many new formulas as possible to formulas that have already become self-evident to the students:

- One classic strategy is to encourage students to create a visual image of the area of a parallelogram in relation to a rectangle, which can enclose the parallelogram when a triangular portion is shifted left or right, and which, therefore, exhibits an equivalent area.
- Another strategy is to illustrate the formula for the area of a circle by cutting a circle into pie-slice wedges and rearranging them into a near-parallelogram, which gives students real reason to believe that the area of a circle really is “pi times radius squared.” (This particular model makes an excellent poster for the classroom and can be an even more effective poster when it is the well-executed work of a classmate.)
- Yet another strategy is to demonstrate how percentages work in retail stores by using mark-ups and discounts as examples. Having different groups of students compute two different scenarios – such as those presented in Example 14 of the Percent Lesson – can be an easy and convincing way for students to understand the algebraic mathematics involved in such situations.

Helping students make sense of mathematical formulas can teach them to “trust” the formulas they are asked to use and cannot yet prove (such as *Ohm’s Law* which relates volts, amps, and resistance).

Aligning with the NCTM Process Standards

The NCTM Process Standards that are addressed in the use and understanding of formulas include: connections, representation, and reasoning and proof. The connections between mathematics and science become readily apparent in this unit as students recognize and apply mathematics in contexts outside of mathematics (CON.3) – e.g. questions about the speed of sound, resistance, voltage, and the time required for a jet plane to climb a given distance. Several formulas illustrate how business and finance use mathematics to relate costs, show profit and loss margins, and calculate interest rates. These formulas also encourage students to recognize and use connections among mathematical ideas in the use of opposite operations to solve problems (CON.1).

As students strive to understand the derivation of various formulas, many connections occur (CON.2). Students use representations repeatedly to model and interpret physical, social and mathematical phenomena (R.3). Conjectures are made and investigated (RP.2), arguments are developed and evaluated (RP.3), and various types of reasoning and methods of proof (RP.4) are utilized.

- CON.1 - Recognize and use connections among mathematical ideas
- CON.2 - Understand how mathematical ideas interconnect and build on one another to produce a coherent whole
- CON.3 - Recognize and apply mathematics in contexts outside of mathematics
- RP.2 - Make and investigate mathematical conjectures
- RP.3 - Develop and evaluate mathematical arguments and proofs
- RP.4 - Select and use various types of reasoning and methods of proof
- R.3 - Use representations to model and interpret physical, social, and mathematical phenomena

3.1 One-Step Equations

Learning Objectives

At the end of this lesson, students will be able to:

- Solve an equation using addition.
- Solve an equation using subtraction.
- Solve an equation using multiplication.
- Solve an equation using division.

Vocabulary

Terms introduced in this lesson:

equal
equation
isolate
linear equation

Teaching Strategies and Tips

Use the introductory problem to motivate equivalent equations, since it can be solved two ways:

- The cost *plus* the change received is *equal* to the amount paid, $x + 22 = 100$.
- The cost is *equal* to the *difference* between the amount paid and the change, $x = 100 - 22$

Examples 1 and 2 are essentially the same; constant terms must be added to both sides to isolate x on the left. Adding vertically can benefit students.

Additional Example.

Solve $12 = -4 + x$.

Solution. To isolate x , add 4 to both sides of the equation. Add vertically.

$$\begin{array}{r} 12 = -4 + x \\ +4 = +4 \\ \hline 16 = x \end{array}$$

The variable in Example 3 is not the usual x . Remind students that the letter of the variable does not matter. Additional Example.

Solve $-21 = n + 14$.

Hint: To isolate n , subtract 14 from both sides of the equation.

In Examples 4-6, teachers may opt to solve the equations by *adding the opposite* in lieu of subtracting.

Example.

Solve $-17 = x + 8$.

Hint: To isolate x , *add* -8 to both sides of the equation. Add vertically.

$$\begin{array}{r} -17 = x + 8 \\ -8 = -8 \\ \hline -25 = x \end{array}$$

Use Example 6 as an example of an equation with fractions. Remind students to find common denominators.

Point out in Example 8 that in general,

$$\frac{ax}{b} = \frac{a}{b}x$$

which will help students isolate x in *one step*, multiplying by the reciprocal of a/b .

Note that the equation in Example 10 can be written in dollars or in cents:

- $5x = 3.25$ (x in *dollars*)
- $5x = 325$ (x in *cents*)

Although Examples 13 and 15 can be solved by making a table and Example 14 by guessing and checking, teachers are encouraged to help students setup and solve an equation of the type presented in the lesson.

Error Troubleshooting

General Tip: After the constant term is canceled in a one-step equation, the variable must be carried down onto the next line. Remind students to write the $x =$.

General Tip: Students forget to perform the same operation on both sides of an equation. Have students use a colored pencil to write what they are doing to both sides of the equation.

3.2 Two-Step Equations

Learning Objectives

At the end of this lesson, students will be able to:

- Solve a two step equation using addition, subtraction, multiplication, and division.
- Solve a two-step equation by combining like terms.
- Solve real-world problems using two-step equations.

Vocabulary

Terms introduced in this lesson:

two-step equations

like terms

combining like terms

unknown

equation in two variables

Teaching Strategies and Tips

Use Example 1 to motivate the process of solving two-step equations:

- Two marbles can be removed from *each* pan first.
- The first step in solving two-step equations is to move the constant *away* from the variable term.
- Whereas it was not possible before, the marbles can now be divided into three groups.
- The second step in solving two-step equations is to isolate the variable by *dividing* by its coefficient.
- It is a small jump for students to write an algebraic expression based on the equality implied by the pans and solve it in an analogous way.

To keep the pans in equilibrium, Example 1 also teaches that, “what is done to one side must be done to the other side”.

Because the solution to Example 2 is negative, the balance strategy of Example 1 will not apply. Use a similar problem in which the solution is positive to demonstrate the balance strategy for *variables buried in parentheses*.

Example.

Six bags each containing the same unknown number of blue marbles and 1 red marble are placed on one side of a balance. 12 red marbles are put on the other side. The scales balance. How many blue marbles are in each bag? Assume the marbles weigh the same and the bags weigh nothing.

Solution. The unknown quantity is *the number of blue marbles in each bag* and is denoted by x . The problem can be summed up as “Six bags of x blue marbles and 1 red equals 12 red marbles” and translated as

$$6(x + 1) = 12$$

This is an example of an equation where x is *buried in parentheses*. To find the number of blue marbles, proceed in one of two ways: (1) Observe that 1 bag weighs the same as 2 red marbles. This is equivalent to dividing both sides of the above equation by 6. (2) Empty the contents of each bag onto the pan. There will be 6 red marbles and 6 times the number of blue marbles that was in one bag. This is equivalent to distributing the 6 in the above equation. In both approaches, the equations are reduced to the familiar one-step and two-step equations, respectively.

The first approach is easier since 12 is evenly divided by 6.

$$\begin{aligned} \frac{6(x + 1)}{6} &= \frac{12}{6} \\ x + 1 &= 2 \\ -1 &= -1 \\ x &= 1 \end{aligned}$$

General Tip: The first step in solving equations with *variables buried in parentheses* depends on:

- Whether the constant is evenly divisible by the coefficient. See Example 2.
- Whether fractions are present. See Examples 3 and 4.

Warm-up to Examples 5 and 6 with exercises similar to the following:

Which of the following pairs of expressions are like terms?

- x and $5x$ (Yes.)
- x and xy (No.)
- x and x^2 (No.)
- -11 and 11 (Yes.)

In Examples 8 and 9, two-variable equations will result. Students substitute one of the givens for one variable to determine the other.

Error Troubleshooting

Watch for the switch from Example 9(ii) to 9(iii), *from Celsius to Fahrenheit*.

Review Question 1c. Remind students to distribute the negative to *both* terms in the parentheses.

Review Question 1g. Hint: Write the s term with a common denominator as $\frac{8}{8}s$.

3.3 Multi-Step Equations

Learning Objectives

At the end of this lesson, students will be able to:

- Solve a multi-step equation by combining like terms events.
- Solve a multi-step equation using the distributive property.
- Solve real-world problems using multi-step equations.

Vocabulary

Terms introduced in this lesson:

Teaching Strategies and Tips

Use this lesson to introduce equations requiring several steps to isolate the variable. Although the examples and exercises are more involved than the previous two lessons, there are no new procedures to learn.

An alternative solution to Example 1 is to have students multiply both sides of the equation by the LCD to clear the fraction first.

Example 2 contains an *x* buried in parentheses. Possible questions to ask students about the way to proceed:

multi-step equation Do you want to avoid a fraction if possible?

- Does distributing first amount to fewer steps?

As students work through more examples, they will recognize the shorter solutions. As an informal rule, *distribute when fractions will result from dividing; otherwise divide*.

- Example: $3(1 - x) = 7$. Dividing both sides by 3 results in a fraction, since 7 is not a multiple of 3. Since fractions are to be avoided, distribute.
- Example: $3(1 - x) = 12$. Since 12 is a multiple of 3, divide. Distributing will result in more steps.
- Example: $\frac{3}{2}(1 - x) = 7$. Fewer fractions result by multiplying both sides of the equation by the reciprocal of $\frac{3}{2}$. Alternatively, clear the fraction by multiplying both sides of the equation by 2.
- Example: $3(1 - x) = \frac{7}{11}$. To avoid having to divide the fraction by 3, distribute first. Alternatively, clear the fraction by multiplying both sides of the equation by 11.
- Example: $\frac{3}{4}(1 - x) = \frac{7}{8}$. First clear the fractions by multiplying both sides of the equation by the LCD.

Error Troubleshooting

In Example 3, remind students to distribute the negative along with the 7; i.e.,

$$\begin{aligned}4(3x - 4) - 7(2x + 3) &= 3 \\ \Rightarrow 4(3x - 4) + (-7)(2x + 3) &= 3 \\ \Rightarrow 12x - 16 - 14x - 21 &= 3\end{aligned}$$

Each set of parentheses needs to be expanded before combining like terms in Examples 3 and 4.

The equation in Example 4 contains both fractions and decimals. Combining like terms is easier by converting the fractions to decimals first. *Decimals do not require a common denominator!* Alternatively, clear the fractions by multiplying both sides of the equation by the LCD, 10. The remaining integers and decimals can be combined.

3.4 Equations with Variables on Both Sides

Learning Objectives

At the end of this lesson, students will be able to:

- Solve an equation with variables on both sides.
- Solve an equation with grouping symbols.
- Solve real-world problems using equations with variables on both sides.

Vocabulary

Terms introduced in this lesson:

collect like terms
rational function

Teaching Strategies and Tips

In Examples 1 and 2, use the balance analogy to motivate solving equations with variables on both sides.

It makes no difference in the answer whether the variable or the constant are moved first in an equation.

- In Example 3, students learn to move the variable term and *not* the constant. This results in the fewest number of steps.
- Whether the final step in solving an equation gives “ $x =$ ” or “ $= x$ ”, the answers will be the same.
- Some teachers may prefer to have students commit to collecting like variable terms on the left of the equal sign and the constant terms on the right for the first several problems.
- Other teachers will encourage students to isolate the variable on the side that is most convenient.
- Essentially students should be seeking the quickest routes to the isolated variable.
- As a class, have students vote on how to solve the equations in the Review Questions.

In Examples 5 and 6, the fraction must be eliminated first by multiplying both sides by the LCD. Then the distribution can take place. Any other way means extra steps.

Additional Example.

$$\text{Solve } 5 + \frac{2x}{3} = \frac{7-x}{6} - 1.$$

Solution. Start by eliminating the fractions; multiply by the LCD.

$$30 + 4x = 7 - x - 6$$

Reminder 1: Multiply *each* term by the LCD. *Do not neglect the -1 on the right or the 5 on the left.*

Reminder 2: The result of multiplying $\frac{2x}{3}$ by the LCD is $4x$ because $\frac{2x}{3} \cdot 6 = \frac{2x}{\cancel{3}}(2 \cdot \cancel{3}) = \frac{2x}{1} \cdot 2 = 4x$. *Do not neglect any “un-cancelled” factors.*

Example 8 consists of many givens. Have students organize the given information before translating into an equation to solve.

Error Troubleshooting

Use Example 7 to show that the *whole* factor $(x + 3)$ is the LCD in the equation. Warn students against separating the two terms in the denominator; i.e.,

$$\frac{14x}{(x+3)} = 7$$
$$\cancel{x} \cdot \frac{14x}{(\cancel{x}+3)} = x \cdot 7$$

Note that Example 7 briefly introduces rational equations. After eliminating the equation of the rational function, it becomes an equation they know how to solve.

3.5 Ratios and Proportions

Learning Objectives

At the end of this lesson, students will be able to:

- Write and understand a ratio.
- Write and solve a proportion.
- Solve proportions using cross products.

Vocabulary

Terms introduced in this lesson:

ratio

unit rate

proportion

cross multiplying

Teaching Strategies and Tips

Use the introductory problem and Example 1 to motivate ratios.

- Have students compare ratios for *value* and *number* of coins in the introductory problem; and the *difference* in price and the *ratio* of prices in Example 1.
- Ask follow-up questions such as:

Which ratio is more useful?

Which ratio does the better job summarizing the situation?

How would you explain to the boy that he is actually getting a bad deal?

Can you find an example of each ratio in daily life?

In Example 1, which measure of comparison is more useful?

If you were selling a copy of the book used, which measure would you use in your ad?

Simplify ratios when possible.

- In the case that the denominator is 1, the ratio is called a *unit rate*.

- Use Example 4 to show how a ratio can be more easily understood when simplified as a unit rate. The ratio is 100 miles to 3.2 gallons; or dividing by 3.2, a ratio of 31.25 miles *per gallon*.
- Possible follow-up questions: *Which ratio is easier to understand? Which is more convenient?*
- Point out that the interpretation of the ratio in Example 1, “*The new book costs 20/13 times the cost of the used book*”, corresponds in fact to the *unit rate* $\frac{20}{13}/1$.

Have students read the ratios aloud in Examples 1-4. For instance, in Example 2, $\frac{4}{3}$ can be read as “4 to 3” and can be written as 4 : 3, $4/3$, or $\frac{4}{3}/1$

Compare and contrast Examples 2 and 3, paying close attention to the use of units.

- In the first ratio, the units *feet* cancel; in the second, the units do not cancel.
- It is worthwhile to point out that different units can be used in a ratio: *miles per hour* (mph), *miles per gallon* (mpg), and *feet per second* (ft/s) are common ratios.

Use Example 6 to introduce proportions by setting the two ratios equal. The difference therefore between a ratio and a proportion is that *ratios are expressions* and *proportions are equations*.

Students learn in Examples 7 and 8 that cross-multiplication is the correct way to solve an equation when the variable is in the denominator of a fraction. In an equation where the variable is in the numerator of the fraction such as $\frac{2x}{3} = \frac{5}{7}$, it is not necessary to cross-multiply.

Error Troubleshooting

General Tip: Have students *label* the givens with proper units in all the proportions they setup, thereby avoiding the common mistake of solving a proportion with unlike units.

- In Example 9, the length units and the time units must be the same.
- The immediate next step for a student who sets up the proportion in Example 9 as $\frac{15 \text{ miles}}{20 \text{ minutes}} = \frac{x \text{ miles}}{7 \text{ hours}}$ is to convert the time units to a common unit.

3.6 Scale and Indirect Measurement

Learning Objectives

At the end of this lesson, students will be able to:

- Use scale on a map.
- Solve problems using scale drawings.
- Use similar figures to measure indirectly.

Vocabulary

Terms introduced in this lesson:

indirect measurement

scale

scale drawings

real distance

scaled distance

similar

similar figures

similar triangles

Teaching Strategies and Tips

This lesson introduces students to the basic mathematical methods used in map-making and scale drawings that rely on ratios and proportions.

In Example 1,

- The scale $1 \text{ cm} = 1 \text{ km}$ tells students that the distance in centimeters on the map is the same number of kilometers on the ground.
- Point out that not all maps have such a simple scale. On those maps, proportions must be setup.
- The basic formula is: $\text{scale} = \frac{\text{known distance on a diagram}}{\text{unknown distance in real life}}$, where the denominator is solved for by cross-multiplying.

In Example 2, have students use a calculator on the square root and remind them to round appropriately.

Examples 3 and 4 use a different form of the basic formula:

- $\text{distance on a diagram} = (\text{distance in real life}) \times (\text{scale})$

Example 5 requires use of a ruler. It is recommended that teachers prepare handouts or overheads in advance. See also *Review Questions 1-3*.

Use Example 5 to point out that a standard centimeter rule will be divided into *tenths* whereas a U.S. inches rule will be divided into *sixteenths*.

Use Examples 6 and 7 to introduce similar figures and indirect measurement. Possible motivating questions: *How can the height of the Eiffel Tower or a lighthouse be determined? How are heights of trees or buildings measured?*

Error Troubleshooting

General Tip: Students may have trouble setting up proportions in problems involving scaled drawings. The following may help:

- The type of ruler (centimeter, inch) used is irrelevant.
- If the scale is provided in the drawing (for example, 1 cm *represents* 183 m), then check that the drawing has not been enlarged or reduced from its original size by comparing the scale with an appropriate ruler. Then use

$$\text{distance on a diagram} = (\text{distance in real life}) \times (\text{scale}).$$

- Otherwise, determine the scale by computing

$$\text{scale} = \frac{\text{known distance on a diagram}}{\text{known distance in real life}}.$$

In Examples 6 and 7, corresponding pairs of triangle sides must correspond in the proportion. To help prevent students from setting up incorrect proportions, have them *name* the triangle sides:

$$\frac{\text{long side}}{\text{short side}} = \frac{\text{long side}}{\text{short side}}$$

3.7 Percent Problems

Learning Objectives

At the end of this lesson, students will be able to:

- Find a percent of a number.
- Use the percent equation.
- Find percent of change.

Vocabulary

Terms introduced in this lesson:

percent
percent equation
rate
total
part
base unit
positive percent change
increase
negative percent change
decrease
mark-up

Teaching Strategies and Tips

Work out exercises like Examples 1-7 and place the solutions in a table similar to the one below. This frees up board space and provides students with a handy reference to study later.

Example.

Complete the table.

TABLE 3.1:

Fraction	Decimal	Percent
$\frac{4}{5}$	0.12	2.5%
		0.01%

TABLE 3.1: (continued)

Fraction	Decimal	Percent
$\frac{3}{8}$	0.001	
$\frac{4}{3}$		

Fraction, decimal, and percent conversions can be summed up as:

- *decimal* \Rightarrow *percent*. Multiply by 100 and affix % symbol. Alternatively, move the decimal point two places right and affix % symbol. See Examples 1 and 2. Additional Example. $1.2 = \frac{1.2 \times 100}{100} = 120\%$.
- *percent* \Rightarrow *decimal*. Divide by 100 and remove % symbol. Alternatively, move the decimal point two places left and remove % symbol. See Examples 3 and 4. Additional Example. $0.003\% = \frac{0.003}{100} = 0.00003$.
- *fraction* \Rightarrow *percent*. Represent the fraction as $x\%$ or $x/100$. Solve for x by cross-multiplying. Alternatively, convert the fraction to a decimal. Then convert decimal to percent as above. See Examples 5 and 6. Additional Example. $\frac{2}{5} = \frac{x}{100} \Rightarrow x = 40$. Therefore, $\frac{2}{5} = 40\%$.
- *percent* \Rightarrow *fraction*. Express percent as a ratio (*per* 100). Reduce fraction. See Example 7. Additional Example. $110\% = \frac{110}{100} = \frac{11}{10} = 1\frac{1}{10}$.
- *fractions* \Rightarrow *decimals*. Divide numerator by denominator. Use calculator if necessary. Example.

$$8 \overline{)1.000} \Rightarrow \frac{1}{8} = 0.125$$

- *decimals* \Rightarrow *fractions*. Place the digits after the decimal under the appropriate power of ten and reduce. Example. $0.225 = \frac{225}{1000} = \frac{9}{40}$.

In Examples 8-11, students setup *percent equations* to find the percent of a given number.

- Convert the value for $R\%$ in the percent equation $R\% \times \text{Total} = \text{Part}$ to a decimal before calculating. *Rate* should be expressed as a decimal in $\text{Rate} \times \text{Total} = \text{Part}$.
- Remind students that *of* means to multiply.

Use Examples 12 and 13 to point out that a *positive percent change* means an *increase* in the quantity, and a *negative change* means a *decrease*.

In Example 14, teachers are encouraged to work out the calculations for *Mark-up*, *Final retail price*, *20% discount*, and *25% discount* as a way to motivate the same calculations done algebraically in the next step.

Additional Example.

Is the order in which we calculate discounts and sales tax significant? In other words, should stores subtract the discount first and then add the tax on the new total or should the total amount be taxed first and then have the discount subtracted from that? or does it matter? Assume the discount is a percent and not a fixed discount.

Solution. Consider an example:

Original price of the item = \$12.50

Discount = 35%

Sales tax rate in the county of purchase = 7.75%

Tax First, Discount Second

$$12.50 + 0.0775(12.50) = 13.47$$

$$13.47 - 0.35(13.47) = 8.76$$

Discount First, Tax Second

$$12.50 - 0.35(12.50) = 8.13$$

$$8.13 + 0.0775(8.13) = 8.76$$

The steps above can be repeated algebraically. Let:

p = original price of the item

d = discount

t = sales tax rate in the county of purchase

Tax First, Discount Second

$p + t(p)$ = amount with tax

$p + t(p) - d(p + t(p))$ = final amount with discount

Discount First, Tax Second

$p - d(p)$ = amount after discount

$p - d(p) + t(p - d(p))$ = final amount with discount

Simplifying the two expressions for the final amount results in identical expressions (careful when distributing).

Conclusion: There is no difference in the total amount to be paid if the tax is added to the total first, followed by the discount, or the discount applied to the total first, followed by the tax.

General Tips:

- Give students time to consider the possibilities on their own.
- Have students choose their own numbers for *original price*, *discount*, and *sales tax rate*. This can be done in groups or individually. Calculators are recommended.
- Asking around the classroom, it is suspicious that everyone's final two calculations are the same. Use this to formulate a conjecture.
- Ask students to turn in their reasoning process as an assignment.
- As the above algebraic argument is completely variable driven (no numbers), teachers are advised to show each step.
- Students reason in various ways. Some common responses are:

"Add the tax first, otherwise you are cheating the government out of its tax."

"Taking the discount first decreases the bill, and so the tax will not be as great."

"Tax should be added on last, as the discounted price is the true price of the item – since that is how much the item is being sold for."

"Doing the tax first *increases* the amount to be paid and so the discount will be larger." According to students, this translates to a smaller price to be paid.

"The quantities must be equal since they have seen it being done in both ways."

As a class, discuss the validity of some of these responses.

3.7. PERCENT PROBLEMS

Error Troubleshooting

Despite the % symbol in Example 4, students see the decimal and incorrectly move it two places right. This error is common for percents less than 1 (i.e., 0.5%) and percents greater than 100 (i.e., 110%).

Additional Examples.

Express 0.01% as a decimal.

Solution: Move the decimal two places left and remove the % symbol. $0.01\% = 0.0001$.

Express 120.25 as a percent.

Solution: Move the decimal two places right and affix the % symbol. $120.25 = 1.2025\%$.

3.8 Problem-Solving Strategies: Use a Formula

Learning Objectives

At the end of this lesson, students will be able to:

- Read and understand given problem situations.
- Develop and apply the strategy: Use a Formula.
- Plan and compare alternative approaches to solving problems.

Vocabulary

Terms introduced in this lesson:

procedure

Teaching Strategies and Tips

In this lesson, students solve applied problems using the four-step problem solving plan.

In Example 2, providing a drawing of each stage of the climb may help students organize the given information.

In Example 4, students can check their answers by working backwards. Consider the fraction remaining after each deduction.

In the *Review Questions*, use the answer to Problem 6 to solve Problem 7.

Error Troubleshooting

General Tip: Remind students to *interpret* their answer in Step 4 of the problem solving plan. This gives students a moment to determine whether or not they've answered the question being asked in the problem.

In Step 4 of Example 3, the discrepancy between the (approximate) solution and the actual time comes from rounding in Step 3.

CHAPTER 4 TE Graphs of Equations and Functions

CHAPTER OUTLINE

- 4.1 THE COORDINATE PLANE
 - 4.2 GRAPHS OF LINEAR EQUATIONS
 - 4.3 GRAPHING USING INTERCEPTS
 - 4.4 SLOPE AND RATE OF CHANGE
 - 4.5 GRAPHS USING SLOPE-INTERCEPT FORM
 - 4.6 DIRECT VARIATION MODELS
 - 4.7 LINEAR FUNCTION GRAPHS
 - 4.8 PROBLEM-SOLVING STRATEGIES - GRAPHS
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Overview

Formulas and equations describe the relationships between quantities. Graphs represent them visually. This chapter emphasizes reading and interpreting graphs.

Suggested pacing:

The Coordinate Plane - 1 hr Graphs of Linear Equations - 1 hr Graphing Using Intercepts - 1 – 2 hrs Slope and Rate of Change - 1 – 2 hrs Graphs Using Slope-Intercept Form - 1/2 hr Direct Variation Models - 1 hr Linear Function Graphs - 1/2 hr Problem-Solving Strategies: *Use a Graph* - 2 hrs

If you would like access to the Solution Key FlexBook for even-numbered exercises, the Assessment FlexBook and the Assessment Answers FlexBook please contact us at teacher-requests@ck12.org.

Problem-Solving Strand for Mathematics

In this chapter, the problem-solving strategy is to *Use a Graph*. The examples and review questions encourage students to *think* about the data presented, to discuss reasonable tolerances for estimates, and to interpret graphs in real-life contexts.

Alignment with the NCTM Process Standards

The NCTM Process Standards in the use of graphs include portions of the communication, connections, and representation standards. One of the key requirements when constructing graphs is to organize and consolidate mathematical thinking in order to display the information accurately (COM.1). Graphing requires recognizing and using connections among mathematical ideas (CON.1) such as independent and dependent variables, appropriate scale when assigning values to the x - and y -axes, or patterns and trends in the data displayed.

- COM.1 - Organize and consolidate their mathematical thinking through communication.
- COM.3 - Analyze and evaluate the mathematical thinking and strategies of others.
- CON.1 - Recognize and use connections among mathematical ideas.
- CON.3 - Recognize and apply mathematics in contexts outside of mathematics.
- R.1 - Create and use representations to organize, record, and communicate mathematical ideas.
- R.3 - Use representations to model and interpret physical, social, and mathematical phenomena.

4.1 The Coordinate Plane

Learning Objectives

At the end of this lesson, students will be able to:

- Identify coordinates of points.
- Plot points in a coordinate plane.
- Graph a function given a table.
- Graph a function given a rule.

Vocabulary

Terms introduced in this lesson:

coordinate plane

origin

quadrant

ordered pair

positive

positive

relation

domain

range

graph of a function

continuous function

discrete function

independent variable

dependent variable

linear relationship

discrete problem

slope

intercept

Teaching Strategies and Tips

Introduction: Motivate xy -coordinates with examples from daily life that employ rectangular coordinate systems.

- Examples: a city map, the game of *Battleship*, a chessboard, spreadsheets, assigned seating at a theater.

- Discuss how to find a particular location in each example: a seat in a theater can be found by row number and then by seat number.
- Point out that the examples are *lattices*, differing from the Cartesian coordinate system in that they are discrete.

Use Examples 1-3 to demonstrate finding coordinates of points on a graph and Examples 4 and 5 to plot points given their coordinates. Allow the class to make observations such as:

- The coordinates of a point cannot be interchanged since the first coordinate specifies going left/right and the second coordinate, up/down. For example, $(2, 7)$ is not the same point as $(7, 2)$.
- If a coordinate of a point is 0, then the point resides on an axis.
- Quadrants can be distinguished by the signs of the coordinates contained in them. For example, a point having coordinates with the signs $(-, +)$ resides in quadrant II. Points with coordinates having signs $(-, -)$ belong to quadrant III.
- In Example 4, it is necessary to display four quadrants so that all points will be visible. The set of points in Example 5 have only positive coordinates; it is convenient therefore to display only the first quadrant. As an informal rule, axes do not need to be extended farther than the largest and smallest x -coordinates and y -coordinates.
- Resize a graph by rescaling the axes. In general, the x and y -axes can be scaled differently. Axis tick marks do not need to be unit increments.

General graphing tips:

- In applied problems, the independent and dependent variables should be distinguished early. Ask:

What quantity is depending on the other?

- In setting up the axes, a suitable scale must be chosen. Ask:

Will the important features of the graph be visible?

Will it be necessary to use different increments along the two axes?

- Constructing tables is a valuable tool. See Examples 6 & 7. Allow students to use the simple inputs, $x = 0, 1, -1, 2$, in their tables when appropriate.

The second method in Example 7 will be returned to in greater detail in a subsequent chapter.

Error Troubleshooting

General Tip: To determine the graph of a linear relationship, no more than two points are needed. Students can be encouraged to plot at least three to ensure no arithmetical errors were made.

4.2 Graphs of Linear Equations

Learning Objectives

At the end of this lesson, students will be able to:

- Graph a linear function using an equation.
- Write equations for, and graph, horizontal and vertical lines.
- Analyze graphs of linear functions and read conversion graphs.

Vocabulary

Terms introduced in this lesson:

formula
equation

Teaching Strategies and Tips

In Example 1:

- The coefficient of x represents the taxi's rate or cost per mile. Ask students to consider how the graph would change if the taxi charged more per mile, then less per mile.
- The constant in the equation is the taxi's base fee. Ask students to interpret the y -intercept (the taxi's base fee).
- Include units on the axes labels; i.e., x (in miles), y (in dollars).
- Do the same for Example 2. Ask students to also interpret the x -intercept (number of years when the debt will be fully paid).

In Examples 1 and 2:

- Students make estimates and predictions from their graphs. Answers will be approximate, even though the equations can be solved exactly. Emphasis should be placed on reading and interpreting the graphs and not solving equations.
- Students make graphs by constructing tables from the given equations. Have them plot at least 3 points for accuracy.
- Teachers may find it useful to show how the equations derive from the given information.

Use Examples 3 and 4 to motivate the equations of horizontal and vertical lines.

- The equations $x = \text{constant}$ and $y = \text{constant}$ are the simplest equations possible.

- They are two-variable equations despite only one variable being present. The coordinate plane must be used to graph them. Often, students will see an equation like $x = 3$ and plot a single point on the x -axis at $(3, 0)$.
- The equation $y = 1$, for example, signifies that the y quantity is fixed at 1 and x is free to take on any value.
- Given a horizontal line, the equation is $y =$ the y -value of any point on the line. Analogously, vertical lines have the equation $x =$ the x -value of any point on the line. This can help with Example 5.

Additional Examples:

What is the equation of the vertical line passing through $(4, 5)$?

Solution: $x = 4$.

Given that a line passes through $(2, -1)$ and is parallel to the x -axis, what is its equation?

Solution: $y = -1$.

Error Troubleshooting

Allow students to distinguish between the input and output variables in Examples 6 and 7 *before estimating*. This way, they will know whether to begin correctly along the horizontal axis or the vertical axis.

In the *Review Questions*, it may help students to rewrite $y = 6 - 1.25x$ in Problem 1c as $y = -1.25x + 6$.

4.3 Graphing Using Intercepts

Learning Objectives

At the end of this lesson, students will be able to:

- Find intercepts of the graph of an equation.
- Use intercepts to graph an equation.
- Solve real-world problems using intercepts of a graph.

Vocabulary

Terms introduced in this lesson:

intercepts

standard form

coefficients

initial cost

Teaching Strategies and Tips

One way to graph lines is by plotting intercepts. Additional questions to ask students in the introduction:

- *Do all lines have x and y -intercepts?*

Solution: No.

- *Which lines have only one intercept?*

Solution: Horizontal and vertical lines.

- *Is it possible for lines not to have intercepts?*

Solution: No, every line must have at least one intercept.

- *Is it possible for lines to have more than 2 intercepts?*

Solution: Yes, but infinitely many intercepts. The lines $x = 0$ and $y = 0$ cross the axes infinitely many times.

In Examples 2a and 2b, have students convert their fractions into decimals before plotting.

Use the *cover-up* method as a quick way to find intercepts algebraically.

In Examples 4 and 5, encourage students to interpret the meaning of the intercepts in context of the problem.

Error Troubleshooting

General Tip: Students will often attempt to find an x -intercept by setting $x = 0$; and a y -intercept by plugging in 0 for y . The opposite is true. To find an intercept, set the *other* variable to 0.

In Example 4, distinguish between the independent and dependent variables first. To find an appropriate scale for the axes, determine the domain of the independent variable next.

In Example 5, although “without spending *more than* \$30” implies an inequality (translates as ≤ 30), solve the problem as if it were an equality (“spending *exactly* \$30”) and then shade the triangular region. For more on systems of inequalities, see chapter *Solving Systems of Equations and Inequalities*.

4.4 Slope and Rate of Change

Learning Objectives

At the end of this lesson, students will be able to:

- Find positive and negative slopes.
- Recognize and find slopes for horizontal and vertical lines.
- Understand rates of change.
- Interpret graphs and compare rates of change.

Vocabulary

Terms introduced in this lesson:

slope

positive slope, negative slope

climbing, descending

lattice points

change

rate of change

per

undefined slope, infinite slope

interpret a graph

velocity

Teaching Strategies and Tips

Use the introduction to motivate the concept of slope. Point out:

- Just as two points determine a unique line, a point and a slope also determine exactly 1 line.
- Viewing the slope as the ratio, $\frac{\text{rise}}{\text{run}}$, is useful. From one point on the line, knowing how to *rise* and *run* brings you to a second point.
- The slope of a line is constant. That is, for any two points on the line, the ratio $\frac{\text{rise}}{\text{run}}$ is the same.
- If $m = \frac{2}{3}$, then rise = 2 and run = 3. Since $\frac{2}{3} = \frac{-2}{-3}$, going *down* two units and then *left* 3 units will also be a point on the line.

Use Example 1 to demonstrate rise-to-run triangles for lines. The triangles are most useful when constructed on lattice points (all coordinates of the vertices are integers). This makes the slope calculation effortless. Observe that the hypotenuse runs along the line.

Use Example 2 to derive a formula for slope.

Emphasize that graphs are read from left to right.

- Linear functions are *increasing* when their graphs slant up and to the right (y increases as x is increased). In this case, slope is positive since Δy and Δx are both positive (or both negative).
- Linear functions are *decreasing* when their graphs slant down and to the right (y decreases as x is increased). In this case, slope is negative since either Δy or Δx is negative, but not both.

Supplement Example 4 with a *skiing* analogy.

- Horizontal lines have *zero slope*, or *no slope*. This corresponds to cross-country skiing.
- Vertical lines have *undefined* slope. This corresponds to falling down a cliff (undefined skiing). Vertical lines have infinite slope

Error Troubleshooting

General Tip. A common mistake is to subtract the x and y -coordinates in different orders in the slope formula; i.e., $m \neq \frac{y_2 - y_1}{x_1 - x_2}$. To avoid making this error, students can write *point 1* and *point 2* above the two points, and then select the coordinates in the same order. See Example 6. Of course, the choice for *point 1* and *point 2* is arbitrary.

4.5 Graphs Using Slope-Intercept Form

Learning Objectives

At the end of this lesson, students will be able to:

- Identify the slope and y -intercept of equations and graphs.
- Graph an equation in slope-intercept form.
- Understand what happens when you change the slope or intercept of a line.
- Identify parallel lines from their equations.

Vocabulary

Terms introduced in this lesson:

slope-intercept form

rise

run

parallel lines

Teaching Strategies and Tips

Use Examples 1 and 2 to make observations such as:

- $m < 0$ when a line slants downward and $m > 0$ when it slants upward.
- $m = 0$ when a line is horizontal.
- $b < 0$ when the y -intercept is below the x -axis and $b > 0$ when it's above the x -axis.
- $b = 0$ when a line passes through the origin.

Use the slope-intercept method to graph lines as an alternative to plotting and joining two intercepts.

With a graphing utility, demonstrate the effects on a line when changing m and b one at a time in an equation in slope-intercept form. Make observations such as:

- The larger the m , the steeper the line.
- Negative slopes can also represent steep lines. The smaller the m (more negative), the steeper the line.
- Slopes approximately equal to zero represent lines that are almost horizontal.
- Changing the intercept shifts a line up/down.
- Parallel lines have the same slope but different y -intercepts.

Error Troubleshooting

In Example 2, use the marked lattice points and/or intercepts in the slope calculation for each line. Using these points allows students to obtain exact answers. See also *Review Questions*, Problems 2 and 3.

4.6 Direct Variation Models

Learning Objectives

At the end of this lesson, students will be able to:

- Identify direct variation.
- Graph direct variation equations.
- Solve real-world problems using direct variation models.

Vocabulary

Terms introduced in this lesson:

direct variation
constant of proportionality
directly proportional

Teaching Strategies and Tips

Use Examples 1-3 to motivate direct variation.

- Direct variation is an expression of a simple linear relationship with y -intercept = 0. Since the line passes through origin, $(0, 0)$ is a point on it.
- The slope of the line is the constant of proportionality and is the only parameter in the problem. Therefore, it takes only one more point to determine the line (to determine the constant of proportionality).
- Emphasize the switch from m (slope) to k (constant of proportionality) is only notational.

Determine the constant of proportionality by substituting the given information (look for one point) into $y = kx$. The result is the equation of variation; use it to answer the question in the problem.

By solving applied problems involving direct variation, teachers demonstrate the usefulness of these models.

Additional Examples:

The cost of raisins varies directly with the number of kilograms of raisins purchased. If the cost is \$34.25 when the number of kilograms purchased is 7.5, calculate the amount of raisins that can be purchased for \$10.50.

Solution: Let C denote the cost of the raisins and N the number of kilograms of raisins purchased. The direct variation equation is $C = k \cdot N$. Substitute $C = 34.25, N = 7.5$.

$$34.25 = k \cdot 7.5$$

$$4.6 \approx k$$

With the constant of proportionality known, calculate N when $C = 10.50$.

$$\begin{aligned}C &= 4.6 \cdot N \\10.5 &= 4.6 \cdot N \\2.2 &= N\end{aligned}$$

So the number of kilograms of raisins that can be purchased for \$10.50 is 2.2 kg.

A ball was thrown from the roof of a high building. The velocity v of the falling ball is directly proportional to the time t of the fall. After 4 seconds, the velocity of the ball is 85 meters per second. What will the velocity be after 6 seconds, assuming it hasn't hit the ground yet?

Hint: Solve for k in the direct variation equation $v = k \cdot t$ first.

Error Troubleshooting

NONE

4.7 Linear Function Graphs

Learning Objectives

At the end of this lesson, students will be able to:

- Recognize and use function notation.
- Graph a linear function.
- Change slope and intercepts of function graphs.
- Analyze graphs of real-world functions.

Vocabulary

Terms introduced in this lesson:

vertical line test
 slope-intercept form
 arithmetic progression
 common difference
 discrete
 continuous

Teaching Strategies and Tips

Students learn to use function notation for the first time.

- $f(x)$ is pronounced “ f of x ”.
- $y = f(x)$, “ y is a function of x ”, and “ y depends on x ” are synonymous. Also, the input or *independent* variable is x .
- $f(x)$ and $y = f(x)$ are interchangeable. This means that the graphing techniques students have learned previously can be used to graph functions.

Point out in Example 1 that the y variable depends on x ; that’s the only reason for solving for y in each case.

Additional Example:

Rewrite the following equation using function notation if cost depends on the number of pounds purchased.

$$C - 3 - 2n = 0$$

Solution: Solve for C and replace C with $C(n)$.

$$C = 2n + 3$$
$$C(n) = 2n + 3$$

Use Example 2 to show how function notation is used. There is nothing new computationally; students evaluate expressions as they before. Remind students to use order of operations.

x -intercepts can be found by solving $f(x) = 0$ and y -intercepts by computing $f(0)$. See Example 4.

Use Example 5 to show how linear functions and arithmetic sequences are related.

- In an arithmetic sequence, terms are found by adding the same constant to the previous term.
- For linear functions, when the input variable is increased by 1, the output variable changes by the value of the slope.

Use Example 5 to distinguish between *discrete* and *continuous*. Use examples from daily life: The number of pumpkin seeds in a pumpkin is discrete. The number of people that can fill a football stadium is discrete. The waiting time for a bus at a bus stop is continuous. The weights of newborn babies are continuous.

Error Troubleshooting

General Tip. $f(x)$ is function notation and should not be confused with multiplication (f times x is not correct).

Example 5c and Problems 6b and 6c of the *Review Questions* have no consecutive terms to subtract to determine the common difference. In such an event, use unknowns for the sequence terms starting with the first given term. Solve the resulting equation. See the hint at the end of Example 5c.

4.8 Problem-Solving Strategies - Graphs

Learning Objectives

At the end of this lesson, students will be able to:

- Read and understand given problem situations.
- Use the strategy: *Read a graph*.
- Develop and apply the strategy: *Make a graph*.
- Solve real-world problems using selected strategies as part of a plan.

Vocabulary

Terms introduced in this lesson:

approximate answer

Teaching Strategies and Tips

Remind the students of the four-step problem-solving plan:

Step 1: Understand the Problem

Read the problem carefully. Once the problem is read, list all the components and data that are involved. This is where you will be assigning your variables.

Step 2: Devise a Plan – Translate

Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart or construct a table as a start to solving your problem.

Step 3: Carry Out the Plan – Solve

This is where you solve the equation you came up with in Step 2.

Step 4: Look – Check and Interpret

Check to see if your answer makes sense.

Emphasize that Step 4 entails more than "going over your work" to see "if it's all there." Cursory checking often misses the error because the same mistake is made again. Above all, encourage students to determine whether their answer seems reasonable.

Error Troubleshooting

General Tip. Have students draw fairly precise graphs to obtain better approximate answers. Use rulers, sharp pencils, graphing paper, and consistent increments on the axes.

CHAPTER 5 TE Writing Linear Equations

CHAPTER OUTLINE

- 5.1 LINEAR EQUATIONS IN SLOPE-INTERCEPT FORM
- 5.2 LINEAR EQUATIONS IN POINT-SLOPE FORM
- 5.3 LINEAR EQUATIONS IN STANDARD FORM
- 5.4 EQUATIONS OF PARALLEL AND PERPENDICULAR LINES
- 5.5 FITTING A LINE TO DATA
- 5.6 PREDICTING WITH LINEAR MODELS
- 5.7 PROBLEM SOLVING STRATEGIES: USE A LINEAR MODEL

Overview

Students apply their knowledge about linear equations to solve real-world problems. They use linear regression methods to fit lines to the data provided and make predictions.

Suggested pacing:

Linear Equations in Slope-Intercept Form - 1 hr Linear Equations in Point-Slope Form - 1 hr Linear Equations in Standard Form - 1 – 2 hrs Equations of Parallel and Perpendicular Lines - 1 – 2 hrs Fitting a Line to Data - 0.5 hrs Predicting with Linear Models - 1 hr Problem Solving Strategies: *Use a Linear Model* - 2 hrs

If you would like access to the Solution Key FlexBook for even-numbered exercises, the Assessment FlexBook and the Assessment Answers FlexBook please contact us at teacher-requests@ck12.org.

Problem-Solving Strand for Mathematics

In this chapter, the problem solving technique *Use a Linear Model* builds directly on lesson material, particularly in “Fitting a Line to Data” and “Predicting with Linear Models.” Within the context of this chapter, linear modeling is defined as using linear interpolation, linear extrapolation, or a line of best fit as a method of predicting trends and/or obtaining reasonable data.

Alignment with the NCTM Process Standards

Being able to approximate or estimate well (R.2) is a valuable skill in mathematics as well as real life. When younger students are asked to estimate, they often follow the rules for rounding rather than truly estimating with regard to the magnitude of a quantity (CON.1). Sometimes teachers unintentionally contribute to this issue because it is difficult to correct estimations; several estimations could be acceptable given different scenarios or different priorities. Taking a few moments to discuss significant digits in real-life situations, such as the cost of a house, a car, or a meal at a restaurant, can really improve students’ number sense and their ability to make appropriate approximations (COM.2; COM.3).

Informal scale drawings can be very helpful whenever geometric shapes are part of a class exercise, and, when done attentively, can internalize the notion of scale (R.1). Free-hand enlargements or miniatures, which many students love to do, can develop an instinct for proportional reasoning (RP.4) and engage artistically inclined students. Displaying attractive, correct student work around the room reinforces the concept of scale and inspires others to think proportionately (R.3).

The question of the reasonableness of a solution is something that must be addressed repeatedly. Teachers should ask students to reflect on the reasonableness of their answers on a regular basis (PS.4).

- COM.2 - Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
 - COM.3 - Analyze and evaluate the mathematical thinking and strategies of others.
 - CON.1 - Recognize and use connections among mathematical ideas.
 - PS.4 - Monitor and reflect on the process of mathematical problem solving.
 - RP.4 - Select and use various types of reasoning and methods of proof.
-
- R.1 - Create and use representations to organize, record, and communicate mathematical ideas.
 - R.2 - Select, apply, and translate among mathematical representations to solve problems.
 - R.3 - Use representations to model and interpret physical, social, and mathematical phenomena.

5.1 Linear Equations in Slope-Intercept Form

Learning Objectives

At the end of this lesson, students will be able to:

- Write an equation given slope and y -intercept.
- Write an equation given the slope and a point.
- Write an equation given two points.
- Write a linear function in slope-intercept form.
- Solve real-world problems using linear models in slope-intercept form.

Vocabulary

Terms introduced in this lesson:

slope-intercept form

Teaching Strategies and Tips

Slope-intercept form:

- Allows quick identification of the slope and y -intercept. This makes graphing linear functions easier, for example.
- Written so that the dependent variable is isolated. This makes finding ordered pairs easier, for example.

Students learn to write linear equations in slope-intercept form, given:

- The slope and y -intercept of the line. See Examples 1, 2, and 7.
- The slope and any one point on the line (b is not given). See Examples 3 and 8.
- Any two points on the line (neither m nor b are given). See Examples 4 and 9.

In Example 2, have students check their answer by choosing two different points on each line and finding the slope. Construct triangles using lattice points.

In Example 4, suggest that students label their ordered pairs, writing x_1, y_1 and x_2, y_2 above each coordinate.

Use Example 5 to demonstrate plugging an expression into a function. Encourage students to keep the parentheses with the expression as it gets plugged into the function.

Additional Example:

Let $f(x) = -2x + 3$. Find $f(-4x + 1)$.

Solution: The entire expression $(-4x + 1)$ must be plugged into the function. Keep the parentheses as you plug in:

$$\begin{aligned}f((-4x + 1)) &= -2(-4x + 1) + 3 \\f((-4x + 1)) &= 8x - 2 + 3 = 8x + 1\end{aligned}$$

Check for student conceptual understanding of function notation. For example,

- $f(-2) = 5$ is equivalent to the ordered pair $(-2, 5)$ or $x = -2$ and $y = 5$.
- $f(0) = 5$ is a y -intercept. $f(3) = 0$ is an x -intercept.

Assume that Examples 7-9 can be modeled by a linear function.

- The slope and y -intercept have contextual significance in applied problems. Have students interpret each in context of the problem.
- In applied problems, students can single out the slope from among the givens by looking for key phrases that signify a slope. For example, *dollars per hour* (Example 7) and *inches per day* (Example 8).

Error Troubleshooting

In Example 2, remind students that a *run* left or a *rise* down are negative quantities. Students commit the error in part a. for example when they go *down* 1, *right* 1 but write the slope as $+1$ instead of -1 .

In Example 3b, remind students to find common denominators.

In Example 4b, remind students to watch the double negative in the denominator: $\frac{3-1}{-2-(-4)}$.

5.2 Linear Equations in Point-Slope Form

Learning Objectives

At the end of this lesson, students will be able to:

- Write an equation in point-slope form.
 - Graph an equation in point-slope form.
 - Write a linear function in point-slope form.
 - Solve real-world problems using linear models in point-slope form.
-

Vocabulary

Terms introduced in this lesson:

point-slope form

Teaching Strategies and Tips

Students learn to write linear equations in point-slope form given:

- The slope and any one point on the line (possibly the y -intercept). See Examples 1, 2, and 8.
- Any two points on the line (m is not given). See Examples 3 and 7.

An equation in point-slope form:

- Uses subscripts on x and y to designate the fixed, given point. x and y assume any other points on the line.
- Is not solved for y . Suggest that students generate other values of y by solving for y first.
- Can be used to graph the line without having to rewrite the equation in slope-intercept form because a slope and a point determine a unique line. See Example 5.

Use Example 3 to show that any point on the line can be substituted for (x_0, y_0) . Point-slope equations will simplify to the *same* slope-intercept equation regardless of the chosen point.

Use Example 6 to introduce function notation for equations in point-slope form.

- Remind students that $f(5.5) = 12.5$ is equivalent to the ordered pair $(5.5, 12.5)$ in $6b$.

“Flat fees”, initial amounts, starting times, etc. correspond to the intercept along the vertical axis.

Error Troubleshooting

In Example 7, have students determine the independent and dependent variables first. This helps them form correct ordered pairs.

5.3 Linear Equations in Standard Form

Learning Objectives

At the end of this lesson, students will be able to:

- Write equivalent equations in standard form.
- Find the slope and y -intercept from an equation in standard form.
- Write equations in standard form from a graph.
- Solve real-world problems using linear models in standard form.

Vocabulary

Terms introduced in this lesson:

standard form

slope

intercepts

Teaching Strategies and Tips

An equation in standard form:

- Can be used to express the equation of a vertical line. This is not possible in slope-intercept or point-slope forms.
- Makes finding intercepts easy. Remind students of the cover-up method introduced in lesson *Graphing Using Intercepts*, chapter *Graphs of Equations and Functions*.

Use Example 3 to find the equations of the lines in standard form using the intercepts and without resorting to slope. The steps are essentially the steps used in the cover-up method only in reverse.

Have students derive the equations that describe the situations in Examples 4 and 5.

Error Troubleshooting

General Tip: Point out that the b in $ax + by = c$ does not represent the y -intercept as it did in an equation in slope-intercept form.

In Example 3c, the x -intercept is a fraction, $x = \frac{3}{2}$. After eliminating the denominator, $2x = 3$, proceed with the method as usual using the 3 instead; i.e. *What number times 3 equals some other number times 4 (the y -intercept)?*

General Tip: Encourage students to include units when labeling their variables.

5.4 Equations of Parallel and Perpendicular Lines

Learning Objectives

At the end of this lesson, students will be able to:

- Determine whether lines are parallel or perpendicular.
- Write equations of perpendicular lines.
- Write equations of parallel lines.
- Investigate families of lines.

Vocabulary

Terms introduced in this lesson:

parallel lines
perpendicular lines
family of lines
vertical shift

Teaching Strategies and Tips

Use the introduction to make observations such as:

- Parallel lines have different y -intercepts.
- Parallel lines have the same slope. Allow students to come to this conclusion using a *rise-over-run* argument: If one line runs (or rises) more than another line, then the lines cannot be parallel; they will eventually meet.
- Perpendicular lines have opposite reciprocal slopes. Encourage students to draw two perpendicular lines. Since one line will be increasing and the other decreasing, this shows that the slopes have opposite signs. Have students also construct the *rise-over-run* triangles for each line. This will demonstrate that the *rise* of one line is the *run* of the other and vice-versa; the slopes are also reciprocal.

Have students recognize in Example 5 that $y = -2$ is the equation of a horizontal line. The problem can then be recast as finding the equation of a vertical line through $(4, -2)$.

Additional Examples:

What is the equation of a line parallel to the x -axis and passing through $(2, -1)$?

Solution: A line parallel to the x -axis is horizontal and therefore has equation, $y = k$. Since it passes through $(2, -1)$, $y = -1$.

Find the equation of a line perpendicular to $x = 3$.

Solution: Since $x = 3$ is a vertical line, a perpendicular line will be horizontal. Therefore any equation of the form $y = k$ is perpendicular to $x = 3$.

Find the equation of a line perpendicular to $x = 3$ that passes through the point $(14, 15)$.

Solution: $y = 15$.

Use Example 9 and a graphing utility to investigate families of lines.

- Fixing the slope m and varying the y -intercept b in $y = mx + b$ results in a set of parallel lines; one through each point on the y -axis.
- Fixing the y -intercept and varying the slope results in all possible lines through that y -intercept; one for every real number angle.

Error Troubleshooting

NONE

5.5 Fitting a Line to Data

Learning Objectives

At the end of this lesson, students will be able to:

- Make a scatterplot.
- Fit a line to data and write an equation for that line.
- Perform linear regression with a graphing calculator.
- Solve real-world problems using linear models of scattered data.

Vocabulary

Terms introduced in this lesson:

scatterplot

measurement error

outlier

linear regression, line of best fit

Teaching Strategies and Tips

Before fitting a line to data, suggest that students always check the scatterplot first for signs of association between the variables. If the points are too scattered, any calculations will be meaningless.

Use the introduction and Example 1 to motivate linear regression.

- Data points are collected from measurement or experiment which is never perfect.
- A line of best fit is the line *closest* to all the data points. This means that the line minimizes the square root of the sum of the squares of the distances from each point to the line.

Provide the class with examples they can relate to. For example:

- Compile a list of students' ages and heights.
- Use a graphing utility to make a scatterplot of *age* vs. *height*.
- Ask the class whether the data set looks approximately linear. If so, perform a linear regression using a graphing utility.
- Use the line of best fit to make predictions of height from ages. Have students gather similar age and *height* data from their older siblings as a way to check the model.

Error Troubleshooting

In Examples 3 and 4, discourage students from using *any* two data points to determine the equation of the line of best fit.

- Two points on the line or two data points *closest* to the line must be used.
- Teachers are encouraged to show that the line of best fit does not always pass through every point in the data set; in fact, it is possible that the line doesn't go through any data point.

5.6 Predicting with Linear Models

Learning Objectives

At the end of this lesson, students will be able to:

- Collect and organize data.
- Interpolate using an equation.
- Extrapolate using an equation.
- Predict using an equation.

Vocabulary

Terms introduced in this lesson:

surveys
experimental measurements
non-linear data
linear interpolation
polynomial interpolation
linear extrapolation
most accurate method

Teaching Strategies and Tips

Use the introduction to motivate data collection and organization.

- Data are gathered from surveys and experimental measurements.
- Data are organized via tables and scatterplots, where it is easier to spot trends and patterns.

In Example 1:

- Point out that two data sets are being displayed simultaneously in the scatterplot. This is a common practice when two data sets are being compared.
- The two variables are *Median Age of Males and Females At First Marriage by Year*.
- Discuss with students whether the scatterplot is approximately linear and whether using a line of best fit to predict future values is appropriate. Do the same for Example 2.

Use Examples 3 and 4 to motivate linear interpolation.

- Ask students how they would go about estimating a value where there is no data point available.

- Possible discussion questions: *Assume the data are linear. How would the line of best fit help? Should only a subset of the data be used? All of the data? How does the above considerations change for non-linear data?*

Use Example 5 to motivate linear extrapolation.

- Point out that the last data point is an outlier and therefore influences the extrapolation heavily.
- Work through the extrapolation a second time using a linear regression. Have students compare answers from the two models.
- Emphasize that extrapolation is not useful when used to predict values far into the future (or far into the past).

For additional data sets, visit:

- <http://www.census.gov>
- <http://www.cdc.gov>

Error Troubleshooting

General Tip: On the TI graphing calculators, students should be using $LinReg(ax + b)$ or $LinReg(a + bx)$ to perform linear regressions and not $LnReg$.

General Tip: Students can neglect to consider the accuracy of a prediction or estimate. Some estimates will not be good because the desired value is far removed from the rest of the data set.

5.7 Problem Solving Strategies: Use a Linear Model

Learning Objectives

At the end of this lesson, students will be able to:

- Read and understand given problem situations.
- Develop and apply the strategy: *Use a Linear Model*.
- Plan and compare alternative approaches to solving problems.
- Solve real-world problems using selected strategies as part of a plan.

Vocabulary

Terms introduced in this lesson:

linear modeling

Teaching Strategies and Tips

In Example 1:

- The problem combines a question from geometry with a solution from algebra.
- Assume a linear association exists between circumference and diameter, which is plausible given the scatterplot.
- Allow students to pick their own points to find the line of best fit. Compare regression models as a class.
- As a class, compute one interpolation and one extrapolation. Compare with the results obtained from the line of best fit.
- All models will contain some error due to measurement. Convince students that the y -intercept should be 0 in any model: a circle of zero diameter has no circumference.

Observe in the scatterplot in Example 2 that the data are *not* linear.

- Assume that the data do not change drastically between the known data points.
- Interpolation or extrapolation methods will give better estimates than the line of best fit.

Error Troubleshooting

General Tip: Before performing a linear regression on a data set, have students always check the scatterplot first for approximately linear data.

CHAPTER 6**TE Graphing Linear Inequalities****CHAPTER OUTLINE**

- 6.1 INEQUALITIES USING ADDITION AND SUBTRACTION**
 - 6.2 INEQUALITIES USING MULTIPLICATION AND DIVISION**
 - 6.3 MULTI-STEP INEQUALITIES**
 - 6.4 COMPOUND INEQUALITIES**
 - 6.5 ABSOLUTE VALUE EQUATIONS**
 - 6.6 ABSOLUTE VALUE INEQUALITIES**
 - 6.7 LINEAR INEQUALITIES IN TWO VARIABLES**
-

Overview

In this chapter, students solve linear inequalities. They use the addition and multiplication properties of inequality and learn that the direction of the inequality is sometimes reversed when solving inequalities. Students solve single-step and multi-step inequalities and progress to compound inequalities and absolute value equations and inequalities. The chapter finishes with graphing inequalities in two variables.

Suggesting Pacing:

Inequalities Using Addition and Subtraction 1 hr

Inequalities Using Multiplication and Division - 1 hr Multi-Step Inequalities - 1 – 2 hrs Compound Inequalities - 1 – 2 hrs Absolute Value Equations - 0.5 hrs Absolute Value Inequalities - 1 hr Linear Inequalities in Two Variables - 0.5 hr

If you would like access to the Solution Key FlexBook for even-numbered exercises, the Assessment FlexBook and the Assessment Answers FlexBook please contact us at teacher-requests@ck12.org.

Problem-Solving Strand for Mathematics

Teachers can increase students' problem solving abilities by presenting them with challenging activities.

A Sudoku warm-up, used on a daily basis for a short period of time or once a week, can lead to excellent conversations on strategies. Solving problems together, students share their thinking and learn from each other. Furthermore, students can continue with Sudoku outside of class and practice various approaches.

Teachers can find Tiling Checkerboard Challenges or other thought-provoking brainteasers on the web and can easily differentiate them to meet the needs of students. Students can make progress with Checkerboard Tiling problems in many different ways. When asked, "Can an ordinary 8×8 checkerboard be covered by 31 dominoes if two squares are removed?" students can sketch solutions, use paper tiles, use graph paper, or, more abstractly, simply consider the two-color arrangements on the checkerboard. There's an entry point for every student, regardless of his or her mathematical background.

Alignment with the NCTM Process Standards

Playing mathematically oriented games such as Sudoku, chess, Hex, Mudcracky, Five-In-a-Row, and Set can incorporate many of the NCTM process standards, particularly the Communication Standard. In a structured classroom setting, students will organize and consolidate their thinking (COM.1), communicate coherently and clearly to peers, teachers, and others (COM.2), and analyze and evaluate the thinking and strategies of others (COM.3). In addition, they will use representations to model and interpret physical phenomena (R.3). Both within and outside the classroom, students will apply and adapt a variety of appropriate strategies to solve problems (PS.3) and select and use various types of reasoning and methods of proof (RP.4).

- COM.1 - Organize and consolidate their mathematical thinking through communication.
- COM.2 - Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
- COM.3 - Analyze and evaluate the mathematical thinking and strategies of others.
- PS.3 - Apply and adapt a variety of appropriate strategies to solve problems.
- RP.4 - Select and use various types of reasoning and methods of proof.
- R.3 - Use representations to model and interpret physical, social, and mathematical phenomena.

6.1 Inequalities Using Addition and Subtraction

Learning Objectives

At the end of this lesson, students will be able to:

- Write and graph inequalities with one variable on a number line.
- Solve an inequality using addition.
- Solve an inequality using subtraction.

Vocabulary

Terms introduced in this lesson:

inequality

interval, interval of values

Teaching Strategies and Tips

Use Examples 1 and 2 to introduce the number line as a way to graph the solution set to an inequality.

- Point out that the solution sets in Examples 1 and 2 represent all numbers which make the statements true. The solution to a linear equation is a *number*; the solution to an inequality, an *interval of (infinite) numbers*.
- Remind students that an open circle is used for inequalities containing a $>$ or $<$ symbol and a closed circle for inequalities containing a \geq or \leq symbol.

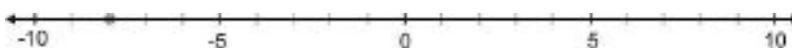
Additional Examples:

Graph the following inequalities on the number line.

- $x < -6$
- $x \geq 3$
- $x > 1$
- $x \leq 10$

Write the inequality that is represented by each graph.

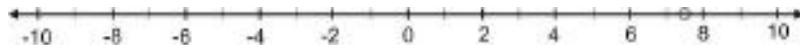
a.



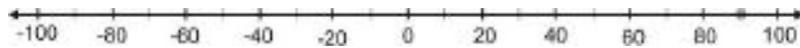
b.



c.



d.



In Example 3, students learn to identify inequalities in sentences. The following chart might be useful for those students having difficulty choosing the correct symbol.

$<$	$>$	\leq	\geq
less than	greater than	at most	at least
fewer	more than	no more than	no less than
		less than or equal to	greater than or equal to

Additional Examples:

Write each statement as an inequality and graph it on the number line.

- You were told not to spend any more than \$20 at the arcade.
- Fewer than 200 tickets are available for sale to the musical performance.
- You must be taller than 40 inches to get on this ride.
- The FDA allows for 30 or more insect fragments per 100 grams of peanut butter.

Use Examples 4 and 5 to show students how to isolate variables in inequalities using addition and subtraction.

- Point out that solving inequalities is analogous to solving equations.
- The exception occurs when multiplying or dividing by a negative number. See the lesson *Inequalities Using Multiplication and Division*.

Error Troubleshooting

General Tip: Suggest to students having difficulty with inequalities that the inequality opens to the larger number.

- $a < b$ is read as “ a is less than b ” or “ b is greater than a ,” depending on the perspective.
- In a statement such as $2 < x$, suggest that students take the point of view of the unknown: “ x is greater than 2” or “all real numbers greater than 2” instead of “2 is less than x ”.

In Example 5d, remind students of *mixed number form* and that $-\frac{3}{4} - 5 = -5\frac{3}{4}$.

6.2 Inequalities Using Multiplication and Division

Learning Objectives

At the end of this lesson, students will be able to:

- Solve an inequality using multiplication.
- Solve an inequality using division.
- Multiply or divide an inequality by a negative number.

Vocabulary

Terms introduced in this lesson:

inequality notation

set notation

interval notation

solution graph

open and closed brackets

including/not including

reversing the inequality

Teaching Strategies and Tips

Use a simple example to present the four ways of expressing the solution set of an inequality:

Solve for x :

$$3x \leq -24$$

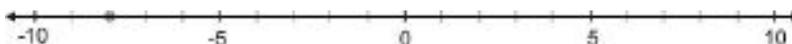
$$\cancel{3}x \leq \frac{-24}{\cancel{3}}$$

$$x \leq -8$$

Solution:

- Inequality notation: $x \leq -8$.
- Set notation: The answer is $\{x|x \text{ a real number, } x \leq -8\}$.

- Interval notation: $(-\infty, -8]$. A combination of parentheses and brackets are used. Because numbers *less than* $-\infty$ are solutions, *negative* infinity is used.
- Solution graph:



The answer is expressed as a closed circle (solid dot) at -8 and shaded to the left.

Tips on set notation:

- The vertical bar separates the variable from the condition that is used to describe the set. It is read as “such that.”
- $\{x|x \text{ a real number, } x \leq -8\}$ is read as “the set of all values of x , *such that* x is a real number less than or equal to -8 .”
- Show students that set notation takes the general form: $\{ \text{“variable” } \textit{such that} \text{ “condition is true” } \}$.
- Students may not see the value of using set notation for the set $\{x|x \text{ a real number, } x \leq -8\}$ when inequality notation suffices. Suggest that although shading a number line or expressing an answer in interval notation may be easier in this example, set notation has advantages in other examples.

Discrete sets are easily described using set notation:

The number of pets belonging to students in class $\{0, 1, 2, 3, 4\}$

The solutions to an equation $\{1, 2\}$.

The set of prime numbers $\{2, 3, 5, 7, 11, 13, 17, \dots\}$.

Tips on interval notation:

- This is the only form of solution that uses the infinity symbol.
- Point out that infinity is always paired with parentheses and never a bracket. The reason for this is because infinity is *not* a number and only a concept. (There is no end to the number line, so it cannot be *included* in the solution set.)
- The two extreme cases of interval notation are (a, a) which represents the single number a , and $(-\infty, \infty)$ which represents all real numbers.

Use Example 2 to show that inequality signs change direction when multiplying or dividing by negative numbers.

Explain why the rule is necessary.

- The number line is constructed so that the negative side is a reflection of the positive side. Multiplying or dividing a number by a negative is equivalent to reflecting it across the origin, as through a mirror.
- For any two positive real numbers on the number line, one will be further to the right than the other. Multiplying or dividing them by a negative has the effect of reflecting them across the origin. The rightmost number on the positive side of the origin becomes the leftmost number on the negative side of the origin.
- Therefore, if a is greater than b , then $-a$ is less than $-b$. This means that the inequality sign changes direction when multiplying or dividing by a negative.
- In the case that one number is negative and the other positive, a simpler argument holds.

Error Troubleshooting

General Tip. Remind students to reverse the inequality sign when multiplying or dividing an inequality by a negative number.

Because the right side of the inequality in Example 2b has a fraction, suggest that students multiply both sides by $-1/9$ to avoid dividing the fraction.

Additional Example:

$$\begin{aligned} -6x &< \frac{2}{9} \\ \left(-\frac{1}{6}\right)(-6x) &< \left(-\frac{1}{6}\right)\left(\frac{2}{9}\right) && \text{The direction of the inequality is changed.} \\ x &> -\frac{1}{27} \end{aligned}$$

In Examples 2b and 2d, remind students that the direction of the inequality does not change on account of the negative sign on the *right* side of the inequality. For example, when solving for x in $12x > [U+0080] [U+0093]30$, do not change the direction of the inequality.

6.3 Multi-Step Inequalities

Learning Objectives

At the end of this lesson, students will be able to:

- Solve a two-step inequality.
- Solve a multi-step inequality.
- Identify the number of solutions of an inequality.
- Solve real-world problems using inequalities.

Vocabulary

Terms introduced in this lesson:

two-step inequality
multi-step inequality
multiple solutions
no solutions
discrete solutions

Teaching Strategies and Tips

In solving multi-step inequalities, remind students to follow the order of operations in each step.

In general, the steps used to solve an inequality are the same as the steps used to solve an equation. The one exception is reversing the inequality sign when multiplying or dividing by a negative.

Use Example 4 to show that an inequality can have a finite and discrete solution set. Compare and contrast this with previous inequalities having an infinite solution set.

Inequalities can have various types of solutions:

- The solution set of $2x \geq 10$ is the *infinite* set $[5, \infty)$.
- The solution set of $12 + x \leq x + 12$ is the *infinite* set $(-\infty, -\infty)$ (all real numbers).
- In the next lesson, students learn that the *infinite* set $[1, 7)$ is a solution set to an inequality.
- The inequality $12 - x \leq -x + 3$ has *no* solutions.
- Inequalities that model real-world problems in which the variables represent integer quantities (usually positive), have *discrete* solution sets. In Example 4, the solution set is $\{0, 1, 2, 3, 4\}$, a *finite, discrete* set. In Examples 5 and 6, the solution sets are *infinite, discrete* sets:

$\{35, 36, 37, \dots\}$ and $\{146, 147, 148, \dots\}$, respectively.

Error Troubleshooting

In Example 3a and Problems 6-10 in the *Review Questions*, remind students to follow the order of operations. Clear parentheses first.

When multiplying both sides of the inequality by 4 in Example 3b, remind students to multiply *both terms* on the right by 4.

In Example 6,

- Remind students that the given numbers must be in the same unit, *dollars* (or *cents*).
- Remind students to round their answers *up* to the next highest integer. Ask them why this is necessary.

Additional Example:

A local bicycle shop advertises bikes for as low as \$235. Ted decides to save his lunch money to purchase one. If he puts away 85 cents daily (including weekends), in how many days will he be able to bring home a bike?

Hint: Convert 85 cents to 0.85 dollars. Round the answer up to 277 days.

6.4 Compound Inequalities

Learning Objectives

At the end of this lesson, students will be able to:

- Write and graph compound inequalities on a number line.
- Solve a compound inequality with “and.”
- Solve a compound inequality with “or.”
- Solve compound inequalities using a graphing calculator (TI family).
- Solve real-world problems using compound inequalities.

Vocabulary

Terms introduced in this lesson:

compound inequalities
combined inequalities

Teaching Strategies and Tips

Use this lesson to show how two or more inequalities can be combined with “and” or “or.”

As a class, discuss the differences between “and” and “or.”

- Use real-world examples:
Earth has one moon *and* Venus has no moons. (True) Earth has two moons *and* Venus has no moons. (False)
Earth has two moons *or* Venus has no moons. (True) Earth has two moons *or* Venus has three moons. (False)
- Allow the discussion to lead to when “and” statements are true and when “or” statements are true.

Introduce compound inequalities in a way similar to single inequalities.

- Have students rewrite compound inequalities as individual inequalities and then solve each one *separately*.
- Solutions to “and” are solutions satisfying *both* inequalities. Solutions to “or” are solutions satisfying *at least one* inequality. See Example 1.
- Suggest that students graph each part of a compound inequality on two separate parallel number lines, aligned at the origin. The solution set can be graphed on a third.

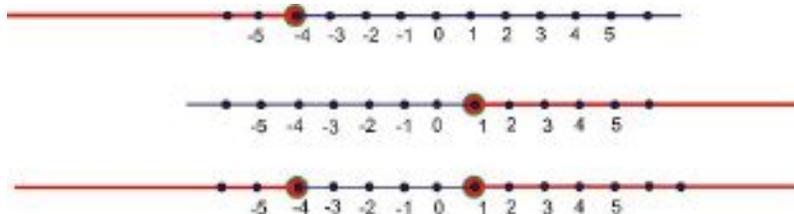
Example:

Graph the following compound inequality on the number line.

$$x \leq -4 \text{ or } x \geq 1.$$

Solution:

Begin by drawing three parallel number lines. On the first, graph the solution set to the first part, $x \leq -4$. On the second, graph the solution set to the second part, $x \geq 1$. Since the statement is joined with an “or”, solutions to the compound inequality, $x \leq -4$ or $x \geq 1$, are those in *at least one* of the two solutions sets.



As a way to check an answer, have students choose one point from each shaded interval. Test these points in the main inequality. Also, test the endpoints to make sure the correct dot was used (solid or hollow).

Use Examples 1a, 1d, 2a, and 2d to introduce compact notation for “and.” For example, $x < 4$ and $x \geq -4$ is rewritten as $-4 \leq x < 4$.

Allow students in Examples 1b, 1c, 2b, and 2c to infer that a solution set consisting of two branches is “or.”

Compound inequalities with “and” can be solved without breaking them into two individual inequalities. See Example 3.

Example:

Solve the compound inequality.

$$-1 < 2x - 3 \leq 7.$$

Solution: *Without* rewriting the compound inequality as two separate inequalities, solve for x by adding 3 to each of the three “sides”.

$$\begin{aligned} -1 + 3 &< 2x - 3 + 3 \leq 7 + 3 \\ 2 &< 2x \leq 10 \end{aligned}$$

Divide each of the three “sides” by 2.

$$\begin{aligned} \frac{2}{2} &< \frac{2x}{2} \leq \frac{10}{2} \\ 1 &< x \leq 5. \end{aligned}$$

As students have trouble with “and” and “or,” encourage them to think of “and” as what’s common to two solution sets. (Ask: What’s common to this interval *and* that interval?) Have them think of “or” as the combination of two intervals. (Ask: What do you get when you put both intervals together?)

6.4. COMPOUND INEQUALITIES

Check for conceptual understanding with exercises such as:

Example:

Graph the following compound inequalities on the number line.

a. $x < -3$ and $x > 2$

b. $x < -3$ and $x < 2$

c. $x > -3$ and $x > 2$

d. $x > -3$ and $x < 2$

e. $x < -3$ or $x > 2$

f. $x < -3$ or $x < 2$

g. $x > -3$ or $x > 2$

h. $x > -3$ or $x < 2$

The exercises above exhaust all the possible cases using “and”, “or”, $[U+0080] [U+009C] < [U+0080] [U+009D]$, $[U+0080] [U+009D] > [U+0080] [U+009C]$ and two arbitrary numbers. Students see the effects of varying one characteristic at a time.

Error Troubleshooting

General Tip. Remind students to use the correct symbols:

TABLE 6.1:

	Interval notation	Inequality notation	Solution graph
Included endpoint	$[,]$	\geq, \leq	solid dots
Excluded endpoint	$(,)$	$>, <$	hollow dots

General Tip. Encourage proper notation.

- An answer such as $-1 < x > -5$ is either incomplete (not simplified) or incorrect. It is unclear whether the student intended on using “and” or “or”.
- In Examples 6 and 7, discourage students from writing $1.875 \geq t \geq 1.56$ and $8.25 \leq t \leq 6.75$. Remind students to reverse the entire expression in their final answer to $1.56 \leq t \leq 1.875$ and $6.75 \leq t \leq 8.25$.

6.5 Absolute Value Equations

Learning Objectives

At the end of this lesson, students will be able to:

- Solve an absolute value equation.
- Analyze solutions to absolute value equations.
- Graph absolute value functions.
- Solve real-world problems using absolute value equations.

Vocabulary

Terms introduced in this lesson:

absolute value
absolute value equations
vertex or cusp

Teaching Strategies and Tips

Focus on the interpretation of absolute value as a distance.

- In Example 1b, $|-120| = 120$ because 120 is 120 units from the origin.
- In Example 2, have students explain, *using a distance argument*, why the order in which the two numbers are subtracted is not important. In general, for any two numbers (or points) a and b , $|a - b| = |b - a|$.
- Have students rethink the simple absolute value equations $|x| = 3$ and $|x| = 10$ in Example 3 as $|\ ? | = 3$ and $|\ ? | = 10$, respectively. (Which numbers are 3 units from the origin? 10 units from the origin?)
- Have students interpret absolute value equations out loud. $|x - 2| = 7$ means “those numbers on the number line 7 units away from 2.” See Examples 4-6. Encourage students to draw the number line and mark the possible solutions.
- Using the distance interpretation, point out that absolute value equations (involving only linear functions) can have no more than 2 solutions. Have students consider absolute value equations with 1 or 0 solutions such as $|x - 2| = 0$ and $|x - 2| = 5$, respectively.

Students have trouble reconciling the definition “ $|x| = -x$ if x is negative” and the fact that “absolute value changes a negative number into its positive inverse.” Offer an example:

- Let $x = 5$. Then $|-5| = -(-5)$ since 5 is negative (using the definition). This simplifies to $|-5| = 5$.

- $|-5|=5$ since absolute value changes a negative number into its positive inverse.

Use Examples 5 and 6 to show students how to rewrite absolute value equations so that the distance interpretation is clearer.

Additional Examples:

a. *Solve the equation and interpret the answer.*

$$|x + 1| = 3$$

Solution: As it stands, the equation cannot be interpreted in terms of distance. Rewrite the equation with a minus sign: $|x - (-1)| = 3$ which can now be interpreted as those numbers 3 units away from -1 . Therefore, the solution set is $\{2, -4\}$.

b. *Solve the equation and interpret the answer.*

$$|3x - 6| = 8$$

Hint: As it stands, the equation cannot be interpreted in terms of distance. Rewrite the equation by dividing both sides by 3:

$$\left| \frac{3x}{3} - \frac{6}{3} \right| = \frac{8}{3}$$

$$|x - 2| = \frac{8}{3}$$

This last equation can now be interpreted as those numbers $8/3$ units away from 2.

Treat the absolute value as a grouping symbol when appropriate.

- The distributive law holds in expressions such as $3|x - 4| = |3x - 12|$ and $\frac{1}{3}|3x - 6| = \left| \frac{3x}{3} - \frac{6}{3} \right| = |x - 2|$.
- The distributive law does not hold in an expression such as $-2|x - 3| \neq |-2x + 6|$
- In general, distribute into absolute value $a \cdot |b| = |a \cdot b|$ when $a = |a|$; i.e., for positive numbers a .
- These steps are based on the property $|a| \cdot |b| = |a \cdot b|$.

When beginning to graph absolute value functions, encourage students to make a table of values such as those in Examples 7 and 8.

Have students plot and describe in words the basic graph $y = |x|$. Allow them to observe the essential properties of the absolute value graph:

- The graph has a [U+0080] [U+009C] V [U+0080] [U+009D] shape, consisting of two rays that meet at a sharp point, called the vertex or cusp.
- One side of the [U+0080] [U+009C] V [U+0080] [U+009D] has positive slope and other side negative slope.
- The vertex is located at the point where the expression inside the absolute value is equal to zero.

Error Troubleshooting

General Tip: Remind students *not* to distribute a negative into the absolute value expression. For example, $-2|x - 3| \neq |-2x + 6|$.

General Tip: Students may misinterpret “absolute value is always positive” and commit the error $-|3 - 5| = 2$. Suggest to students that in such situations $-|3 - 5| = (-1)|3 - 5|$. The multiplication by -1 happens *after* the absolute value has been performed, and so $-|3 - 5| = (-1)|3 - 5| = (-1)|-2| = (-1)2 = -2$.

6.6 Absolute Value Inequalities

Learning Objectives

At the end of this lesson, students will be able to:

- Solve absolute value inequalities.
- Rewrite and solve absolute value inequalities as compound inequalities.
- Solve real-world problems using absolute value inequalities.

Vocabulary

Terms introduced in this lesson:

absolute value inequality

Teaching Strategies and Tips

Use Example 1 to show that the distance interpretation equally applies to absolute value *inequalities*.

Additional Examples:

a. *Solve the inequality.*

$$|x| \leq 10.$$

Solution: $|x| \leq 10$ represents all numbers whose distance from the origin is *less than or equal to* 10. This means that $-10 \leq x \leq 10$.

b. *Solve the inequality.*

$$|x| \geq 10.$$

Solution: $|x| \geq 10$ represents all numbers whose distance from the origin is *greater than or equal to* 10. This means that $x \leq -10$ or $x \geq 10$.

Use Example 1 and the distance interpretation to motivate solving the absolute value inequalities in Examples 2-5.

- Allow students to infer from Example 1 that $|x| < a \Leftrightarrow -a < x < a$ and $|x| > a \Leftrightarrow x < -a$ or $x > a$.

In Problems 4 and 5 in the *Review Questions*, have students divide by the coefficient of x first.

Error Troubleshooting

NONE

6.7 Linear Inequalities in Two Variables

Learning Objectives

At the end of this lesson, students will be able to:

- Graph linear inequalities in one variable on the coordinate plane.
- Graph linear inequalities in two variables.
- Solve real-world problems using linear inequalities.

Vocabulary

Terms introduced in this lesson:

dashed line/solid line

Teaching Strategies and Tips

In this lesson, students learn to graph linear inequalities on the coordinate plane.

- Students draw dashed lines for the strict inequalities ($<$, $>$); interpret this as excluding the points on the line from the solution set. Students draw solid lines for the inequalities \leq , \geq ; interpret this as including the points on the line in the solution set.
- Have students shade those regions of the plane which satisfy the given inequalities.
- As a general rule, shade above the line $y = mx + b$ if the stated inequality is $y \geq mx + b$. Shade below the line if $y \leq mx + b$. Have students solve for y first. See Examples 5-7.
- Graphing inequalities on the coordinate plane is a step up from graphing on the number line and requires more care.

Use Examples 1 and 2 to motivate graphing the *absolute value* inequalities in Examples 3 and 4.

- Have students rewrite the absolute value inequality as a compound inequality first.

In Examples 8 and 9 and Problems 13 and 14 in the *Review Questions*,

- Point out that quadrant I is the only quadrant used because the variables should be positive.
- Only the points with integer coordinates are possible solutions in Example 9.

Error Troubleshooting

General Tip. Students can misinterpret an inequality such as $x > 2$ as an inequality in one variable and incorrectly shade that part of the x -axis for which $x > 2$.

CHAPTER 7

TE Solving Systems of Equations and Inequalities

CHAPTER OUTLINE

- 7.1 LINEAR SYSTEMS BY GRAPHING**
 - 7.2 SOLVING LINEAR SYSTEMS BY SUBSTITUTION**
 - 7.3 SOLVING LINEAR SYSTEMS BY ELIMINATION THROUGH ADDITION OR SUBTRACTION**
 - 7.4 SOLVING SYSTEMS OF EQUATIONS BY MULTIPLICATION**
 - 7.5 SPECIAL TYPES OF LINEAR SYSTEMS**
 - 7.6 SYSTEMS OF LINEAR INEQUALITIES**
-

Overview

In this chapter, students discover the methods that determine solutions to systems of linear equations and inequalities. Students begin by solving systems of equations graphically, realizing that the solution for each system is the point of intersection between the two lines represented by the given equations.

Suggested Pacing:

Linear Systems by Graphing - 1 hr Solving Linear Systems by Substitution - 1 hr Solving Linear Systems by Elimination through Addition or Subtraction - 1 hr Solving Systems of Equations by Multiplication - 1 – 2 hrs Special Types of Linear Systems - 0.5 hr Systems of Linear Inequalities - 2 hrs

If you would like access to the Solution Key FlexBook for even-numbered exercises, the Assessment FlexBook and the Assessment Answers FlexBook please contact us at teacher-requests@ck12.org.

Problem-Solving Strand for Mathematics

In this chapter, *Systems of Equations and Inequalities*, several different methods of solving equations and inequalities are taught. Throughout, real-world problems are incorporated into the exercises, and practical applications of important concepts such as constraints, optimum solutions, feasibility regions, and maximum and minimum values are outlined.

In this chapter you might try giving students limited choices about which problems to do independently. This allows you to assess which of the problems students feel comfortable with, which they wish to avoid, and where some re-teaching might be helpful. With the section *Comparing Methods of Solving Linear Systems*, students might be asked to use all three primary methods on a single exercise rather than doing three different exercises with the same method. In addition to giving necessary practice it may help students become more discerning about which method is simpler in a given situation.

Alignment with the NCTM Process Standards

Using the principle of choice and, occasionally, asking students to write about a problem of their preference can address many of the NCTM Process Standards. Chief among them would be the Communications and Connections standards. Students solving and presenting solutions for equation and inequality problems must organize and consolidate their mathematical thinking (COM.1), communicate their mathematical thinking coherently and clearly to peers, teachers, and others (COM.2), and use the language of mathematics to express mathematical ideas precisely (COM.4). When students share their various insights and approaches to problems as common classroom practice, they learn to analyze and evaluate the mathematical thinking and strategies of others (COM.3).

When students are allowed to select specific problems as part of an assignment rather than to merely mimic the sample problems given in the text, they strive to recognize and use connections among mathematical ideas (CON.1) and to understand how mathematical ideas interconnect and build upon one another to produce a coherent whole (CON.2). They are encouraged to apply and adapt a variety of appropriate strategies to solve problems (PS.3) and use representations to model and interpret physical, social, and mathematical phenomena (R.3).

- COM.1 - Organize and consolidate their mathematical thinking through communication.
- COM.2 - Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
- COM.3 - Analyze and evaluate the mathematical thinking and strategies of others.
- COM.4 - Use the language of mathematics to express mathematical ideas precisely.
- CON.1 - Recognize and use connections among mathematical ideas.
- CON.2 - Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
- PS.3 - Apply and adapt a variety of appropriate strategies to solve problems.
- R.3 - Use representations to model and interpret physical, social, and mathematical phenomena.

7.1 Linear Systems by Graphing

Learning Objectives

At the end of this lesson, students will be able to:

- Determine whether an ordered pair is a solution to a system of equations.
- Solve a system of equations graphically.
- Solve a system of equations graphically with a graphing calculator.
- Solve word problems using systems of equations.

Vocabulary

Terms introduced in this lesson:

system of equations

solution to an equation

solution to a system of equations

point of intersection

Teaching Strategies and Tips

Present students with a basic problem to motivate *systems* of equations.

Example:

Find two numbers, x and y , such that their sum is 10 and their difference is 4.

Allow students some time to find the numbers. Encourage guess-and-check at first. A good place to start is with pairs of integers.

Solution:

The problem can be translated as:

$$x + y = 10$$

$$x - y = 4$$

Ask: Of all the possible ordered pair solutions to the first equation, which also satisfy the second?

x	y	sum	difference
5	5	10	0
6	4	10	2
7	3	10	$4\sqrt{}$

Additional Example:

Complete the table for each equation. Compare the rows of the two tables to determine the solution to the system.

$$2x + y = 8$$

$$x - y = 1$$

x	y
0	
1	
2	
3	

x	y
0	
1	
2	
3	

Hint: Solve each equation for y first.

Use the introduction and Example 1 to point out that a system of equations is *one* problem despite there being *two* equations.

- The two equations must be solved “together” or *simultaneously*.
- The problem is not done until *both* x and y have been determined.
- The solution to an equation is a number; the solution to a system of equations is an ordered pair.
- An ordered pair solution *satisfies*, or “makes the equations true.”

Additional Examples:

Find the solution to the following systems of equations by checking each of the choices in the list.

a.

$$x + y = 3$$

$$2x + y = 1$$

i. (2, 1)

7.1. LINEAR SYSTEMS BY GRAPHING

ii. $(5, -2)$

iii. $(-2, 5)$ ✓

b.

$$\begin{aligned}7x + y &= 7 \\ -3x - 2y &= -14\end{aligned}$$

i. $(1, 0)$

ii. $(0, 7)$ ✓

iii. $(4, 3)$

Use Examples 2-4 to demonstrate the graphing method for solving a system of equations.

- Lines can be graphed using any method: constructing a table of values, graphing equations in slope-intercept form, solving for and plotting the intercepts.

Emphasize that the graphing method approximates solutions.

- It is exact when the point of intersection has integer coordinates or easily discernible rational numbers.
- Suggest that students draw careful graphs.
- By zooming in, a calculator provides the coordinates of the intersection point to any degree of accuracy although the solution can still be approximate.

Error Troubleshooting

General Tip: To generate y values, as for a table, have students solve each equation for y first. See Example 6.

General Tip: To demonstrate that an ordered pair is a solution to a system, remind students that it must satisfy *both* equations. To demonstrate that an ordered pair is *not* a solution to a system, remind students that *at least one* of the equations will not be satisfied.

7.2 Solving Linear Systems by Substitution

Learning Objectives

At the end of this lesson, students will be able to:

- Solve systems of equations in two variables by substituting for either variable.
- Manipulate *standard form* equations to isolate a single variable.
- Solve real-world problems using systems of equations.
- Solve mixture problems using systems of equations.

Vocabulary

Terms introduced in this lesson:

substitution

substitution method

standard form of a linear equation

Teaching Strategies and Tips

Use Example 2 to motivate the substitution method.

- As the solution consists of fractions, the system is more complicated than any example presented up to this point.
- Emphasize that the graphing method can only provide an approximation. Therefore a different method is needed.

The substitution method:

- An algebraic method; provides exact solutions.
- A technique for replacing an unknown with another expression to obtain a third equation with only one unknown (replacing *equals with equals*).
- Best used when one of the coefficients of the variables is 1.

Encourage students to isolate the variable with a coefficient of 1 or -1 . Students often give themselves extra work by choosing an equation and a variable at random.

Mixture problems:

- Mixtures do not necessarily pertain to chemistry. See Example 4 and *Review Question 7*.
- Approach Example 7 with a picture. By the labeling the unknowns in it, the system of equations will be evident.

Error Troubleshooting

In Example 2, have students back-substitute x into one of the *original* equations in case that an error was made in solving for x .

General Tip: In fact, *any* of the two original equations can be used to find y once x is determined. The easier-looking the equation the better.

Point out in Example 3 that the question can be answered after determining $x = 117.65$. There is no need to back-substitute to find the *cost per month*.

General Tip: Remind students to write coordinates in correct order, depending on how they labeled the variables in the beginning.

- Students often incorrectly write the value they found first as x .
- For problems where variables other than x and y are used, have students clearly state which variable is independent and which is dependent. See *Review Questions 5-9*.

7.3 Solving Linear Systems by Elimination through Addition or Subtraction

Learning Objectives

At the end of this lesson, students will be able to:

- Solve a linear system of equations using elimination by addition.
- Solve a linear system of equations using elimination by subtraction.
- Solve real-world problems using linear systems by elimination.

Vocabulary

Terms introduced in this lesson:

elimination
method of elimination
coefficient

Teaching Strategies and Tips

Use Example 1 to motivate the elimination method.

- To find the cost of one banana, it makes sense to subtract the equations.
- In general, equations can be added or subtracted so that a variable cancels.

Show that in Example 2, equations can be added column-wise; each column representing a different variable. When x or y cancel, use $0x$ or $0y$ as placeholders.

In Example 3,

- Drawing a picture helps.
- Encourage students to label variables first. The two unknowns are the *speed of the river* and the *speed of the canoe in still water*.
- Compare going downstream to walking on a moving walkway in an airport terminal. Travelers walk faster than usual as their speeds are boosted by the walkway. The opposite is true going upstream and against the moving walkway.

Error Troubleshooting

Remind students in *Review Problem 3* to align the variables column-wise.

In *Review Problems 4, 6, 7, and 9*, encourage students to put the equation being subtracted in parentheses. This way, it will be easier to remember to distribute the negative.

General Tip: When subtracting two equations, remind students not to forget to subtract the constants on the right side.

7.4 Solving Systems of Equations by Multiplication

Learning Objectives

At the end of this lesson, students will be able to:

- Solve a linear system by multiplying one equation.
- Solve a linear system of equations by multiplying both equations.
- Compare methods for solving linear systems.
- Solve real-world problems using linear systems by any method.

Vocabulary

Terms introduced in this lesson:

scalar

lowest common multiple

Teaching Strategies and Tips

Use the introduction to convince students that the elimination method applies to *any* linear system because one or both equations can be multiplied by a constant resulting in a “new” pair of equations with matching coefficients.

In Example 1, some students have trouble keeping track of the multipliers for each equation. Try using a visual:

$$\begin{array}{ccc}
 7x + 4y = 17 & \text{same} & 7x + 4y = 17 \\
 5x - 2y = 11 & \xrightarrow{x^2} & 10x - 4y = 22
 \end{array}$$

In Example 2,

- Point out that the distance covered is the same in both directions, so the 400 yards is unnecessary information.
- Remind students to back-substitute to complete the problem.

Use Example 3 to demonstrate a system with no matching coefficients and no coefficients that are multiples of others.

- Remind students how to find the LCM of two numbers.

- Suggest that students find the LCM of the “smaller pair” of coefficients, 3 and 5, instead of 880 and 1845.

Teachers may decide to forgo back-substitution and instead teach elimination of the second variable (“double elimination”).

Additional Example:

Solve the system using multiplication.

$$\begin{aligned} 2x - 5y &= 8 \\ -11x + 3y &= 15 \end{aligned}$$

Solution by “double elimination”. Eliminate x first.

$$\begin{array}{rcl} 2x - 5y = 8 & \xrightarrow{\times 11} & 22x - 55y = 88 \\ -11x + 3y = 15 & \xrightarrow{\times 2} & -22x + 6y = 30 \end{array}$$

Add.

$$\begin{array}{r} 22x - 55y = 88 \\ -22x + 6y = 30 \\ \hline 0x - 49y = 118 \end{array}$$

Divide by -49 . Therefore, $y = -\frac{118}{49}$.

To find x , eliminate y next.

$$\begin{array}{rcl} 2x - 5y = 8 & \xrightarrow{\times 3} & 6x - 15y = 24 \\ -11x + 3y = 15 & \xrightarrow{\times 5} & -55x + 15y = 75 \end{array}$$

Add.

$$\begin{array}{r} 6x - 15y = 24 \\ -55x + 15y = 75 \\ \hline -49x + 0y = 99 \end{array}$$

Divide by -49 . Therefore, $x = -\frac{99}{49}$.

Answer: The solution to the system is $(-\frac{99}{49}, -\frac{118}{49})$.

The advantage of “double elimination” is that the fraction does not need to be back-substituted.

Error Troubleshooting

General Tip: Students forget to multiply *every* term in an equation by the scalar.

General Tip: Encourage students to eliminate the variable whose coefficients in both equations have the smallest LCM.

Remind students in *Review Problem 2e* to align the variables column-wise.

7.5 Special Types of Linear Systems

Learning Objectives

At the end of this lesson, students will be able to:

- Identify and understand what is meant by an inconsistent linear system.
- Identify and understand what is meant by a consistent linear system.
- Identify and understand what is meant by a dependent linear system.

Vocabulary

Terms introduced in this lesson:

consistent system

inconsistent system

infinite number of solutions

dependent system

determining the system

Teaching Strategies and Tips

Use this lesson to classify systems of equations according to the number of solutions they have.

- Encourage students to use the graphical interpretation and rewrite equations in slope-intercept form first to compare slopes. See Examples 1-4.

Use Examples 5 and 6 to show how to classify a system using the substitution or elimination methods.

Solving inconsistent or dependent systems by substitution or elimination leads to variables dropping-out.

- A false statement, such as $3 = 4$, indicates an inconsistent system (parallel lines).
- A true statement, such as $-12 = -12$, indicates dependent system (coinciding lines).

When students arrive at an answer such as $8 = 8$, they may assign 8 to one of the variables and solve for the other variable.

Use Example 6 to point out that the equations in a dependent system are multiples of each other.

Additional Example:

Solve the system by multiplication.

$$\begin{aligned} 3x &= -y - 5 \\ 2y + 10 &= -6x \end{aligned}$$

Solution: Align the variables column-wise.

$$\begin{aligned} 3x + y &= -5 \\ 6x + 2y &= -10 \end{aligned}$$

Eliminate y by multiplying the first equation by -2 .

$$\begin{array}{rcl} 3x + y = -5 & \xrightarrow{x(-2)} & -6x - 2y = 10 \\ 6x + 2y = -10 & \text{same} & 6x + 2y = -10 \end{array}$$

Add.

$$\begin{array}{r} -6x - 2y = 10 \\ 6x + 2y = -10 \\ \hline 0x + 0y = 0 \end{array}$$

Therefore, the system is dependent. Notice that the equations are multiples of one other: $3x + y = -5 \xrightarrow{\times 2} 6x + 2y = -10$.

In Examples 7-9, emphasize the geometric interpretation in context:

- In Example 7, the lines intersect because the rental fees are different for the two membership options.
- In Example 8, the lines are parallel because the memberships have different flat fees (y -intercepts), but the same rental fee (slope).
- In Example 9, the lines coincide because the equations use the same information. It is not possible to determine the price of each fruit since the second equation does not give any new information.

Error Troubleshooting

General Tip: In dependent systems, students sometimes misinterpret “an infinite number of solutions” to mean that *any* ordered pair will satisfy the given system of equations. Of course, only the infinite number of points *on the line* that the equations represent are solutions.

7.6 Systems of Linear Inequalities

Learning Objectives

At the end of this lesson, students will be able to:

- Graph linear inequalities in two variables.
- Solve systems of linear inequalities.
- Solve optimization problems.

Vocabulary

Terms introduced in this lesson:

system of inequalities

half-plane

dotted line/ solid line

bounded solution/ unbounded solution

linear programming

constraints

feasibility region

optimization equation

maximum/minimum value

Teaching Strategies and Tips

Have students follow Example 1 step-by-step for the first few *Review Questions*.

- Shade each region differently.

Encourage students to rewrite each equation in slope-intercept form. This will help them graph the line and decide which half-plane to shade.

Use Example 2 as an illustration of a system of inequalities with no solution.

- Because the lines are parallel, the shaded regions will never intersect.
- It is possible, however, for lines to be parallel and have shaded regions intersect. For instance, reverse the inequalities in Example 2.

In Example 3,

- Emphasize that the method used to determine solutions to a system of inequalities can be extended to *any* number of inequalities.
- Point out that the pair of inequalities, $x \geq 0$ and $y \geq 0$ describes the first quadrant of the coordinate plane. In fact, any quadrant can be similarly described:

Quadrant I: $x \geq 0$ and $y \geq 0$

Quadrant II: $x \leq 0$ and $y \geq 0$

Quadrant III: $x \leq 0$ and $y \leq 0$

Quadrant IV: $x \geq 0$ and $y \leq 0$

Have students follow Examples 5 and 6 step-by-step for *Review Question 8*.

Error Troubleshooting

General Tip: Remind students to reverse the direction of the inequality sign when multiplying or dividing by a negative number. See *Review Questions 1-7*.

CHAPTER 8**TE Exponential Functions****CHAPTER OUTLINE**

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- 8.1 EXPONENT PROPERTIES INVOLVING PRODUCTS**
 - 8.2 EXPONENT PROPERTIES INVOLVING QUOTIENTS**
 - 8.3 ZERO, NEGATIVE, AND FRACTIONAL EXPONENTS**
 - 8.4 SCIENTIFIC NOTATION**
 - 8.5 EXPONENTIAL GROWTH FUNCTIONS**
 - 8.6 EXPONENTIAL DECAY FUNCTIONS**
 - 8.7 GEOMETRIC SEQUENCES AND EXPONENTIAL FUNCTIONS**
 - 8.8 PROBLEM-SOLVING STRATEGIES (REPRISE MAKE A TABLE; LOOK FOR A PATTERN)**
-

Overview

Students learn to simplify expressions involving exponents. They work with scientific notation and then move on to exponential growth and decay and finally geometric sequences.

Suggested Pacing:

Exponent Properties Involving Products - 1 hr Exponent Properties Involving Quotients - 1 hr Zero, Negative, and Fractional Exponents - 0.5 hr Scientific Notation - 1 hr Exponential Growth Functions - 1 hr Exponential Decay Functions - 1 hr Geometric Sequences and Exponential Functions - 1 – 2 hrs Problem-Solving Strategies: reprise *Make a Table; Look for a Pattern* - 2 hrs

If you would like access to the Solution Key FlexBook for even-numbered exercises, the Assessment FlexBook and the Assessment Answers FlexBook please contact us at teacher-requests@ck12.org.

Problem-Solving Strand for Mathematics

This chapter, *Exponential Functions*, begins with the essentials of exponential notation and builds sequentially through fairly sophisticated uses of exponents to solve real-world problems. The patterns related to exponential notation provide an opportunity to demonstrate the power of definitions. Looking for a pattern and using a table for projecting compound interest, for example, allows the repeated factor to be discovered and affirms that an exponent is used appropriately in the formula.

Alignment with the NCTM Process Standards

NCTM Process Standards from every strand can be seen in the lesson Problem-Solving Strategies. Students will both analyze and evaluate their own mathematical thinking and the strategies of others (COM.3) and use the language of mathematics to express mathematical ideas precisely (COM.4). In tackling business and scientific real world problems, they will make connections, recognize and apply mathematics in contexts outside of mathematics (CON.3). From a problem-solving perspective students will apply and adapt a variety of appropriate strategies

to solve problems that arise in mathematics and in other contexts (PS.2, PS.3), and they will recognize reasoning and proof—specifically in applying the use of defining terms—as fundamental aspects of mathematics (RP.1). In developing tables to help them formulate their conclusions, students will create and use representations to organize, record and communicate mathematical ideas (R.1) and use these representations to model and interpret physical, social, and mathematical phenomena (R.3). This is a rich problem-solving lesson indeed!

- COM.3 - Analyze and evaluate the mathematical thinking and strategies of others.
- COM.4 - Use the language of mathematics to express mathematical ideas precisely.
- CON.3 - Recognize and apply mathematics in contexts outside of mathematics.
- PS.2 - Solve problems that arise in mathematics and in other contexts.
- PS.3 - Apply and adapt a variety of appropriate strategies to solve problems.
- RP.1 - Recognize reasoning and proof as fundamental aspects of mathematics.
- R.1 - Create and use representations to organize, record, and communicate mathematical ideas.
- R.3 - Use representations to model and interpret physical, social, and mathematical phenomena.

8.1 Exponent Properties Involving Products

Learning Objectives

At the end of this lesson, students will be able to:

- Use the product of a power property.
- Use the power of a product property.
- Simplify expressions involving product properties of exponents.

Vocabulary

Terms introduced in this lesson:

power

exponent

square, cube

base

factors of the base

product rule for exponents

power of a product

power rule for exponents

Teaching Strategies and Tips

Encourage students to review basic exponents in the chapter *Real Numbers*.

- An exponent is a notation for repeated multiplication of a number, variable, or expression.
- Exponents count how many *bases* there are in a product
- Parentheses precede exponents in the order of operations.

Additional Examples:

Write in exponent form.

a. $(4)(4)(4)$

b. $(-7)(-7)(-7)(-7)$

c. $(5w)(5w)(5w)$

d. $t \cdot t \cdot t \cdot t \cdot t$

Encourage proper use of mathematical language:

- 3^4 is read as “three to the fourth power,” “three to the fourth,” or “three raised to the power of four.” The exponents 2 and 3 are special: 3^2 , “three squared” and 3^3 , “three cubed.”
- $(3x)^3$ is read as “quantity $3x$ cubed.”
- -3^2 is read as “opposite of *three squared*.”
- $(-3)^2$ is read as “*negative three squared*”.

To check for conceptual understanding, ask students to translate the following into symbols.

- Square negative two. Answer: $(-2)^2$
- Negative two squared. Answer: $(-2)^2$
- The opposite of two squared. Answer: -2^2

Allow the class to infer the product rule for exponents in Example 2.

- “When you multiply like bases you *add* the exponents.”

Allow the class to infer the power rule for exponents in Example 6c.

- “When you raise an exponent to an exponent, you multiply them.”

Combine exponent rules in Examples 6-9.

- Suggest that students apply each rule one step at a time.
- Order of operations must be followed at each step: evaluate exponents before multiplying. See Examples 6a, 6b, and 9.

Additional Examples:

Simplify the following expressions.

a. $(4x)(3x)^2$

Hint: Apply the exponent before multiplying.

b. $(5x^2 + (-2x)^2)^3$

Hint: Simplify in the parentheses first.

c. $(4x)(3x)^2 + (5x^2 + (-2x)^2)^3$

Hint: Simplify each term first.

Point out in Example 8 where the commutative property of multiplication is being used.

Error Troubleshooting

Use Examples 4 and 5 to point out two common errors:

- Multiplying bases incorrectly (exponents are different). $2^4 \cdot 2^3 \neq 4^7$.
However, $3^4 \cdot 2^4 = 6^4$ because $3^4 \cdot 2^4 = (3 \cdot 2)^4 = 6^4$ (exponents are same).
- Applying the product rule incorrectly (bases are different). $3^4 \cdot 2^3 \neq 6^7$.

8.1. EXPONENT PROPERTIES INVOLVING PRODUCTS

Remind students in *Review Question 6* that even powers of negative numbers are *positive*. Try using a visual to show that negatives cancel in pairs:

$$(-2)^6 = (-2)(-2)(-2)(-2)(-2)(-2) = \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} = 64$$

General Tip: Occasionally, students forget that:

- negative \times positive = negative
- negative \times negative = positive

In *Review Question 13*, remind students that the exponent in $-2y^4$ does not apply to -2 . Therefore, $-2y^4$ is simplified.

Additional Examples:

- Explain the difference between $4x^2$ and $(4x)^2$.
- Explain the difference between -1^2 and $(-1)^2$.

Teachers are encouraged to survey the class for answers. Ask: What constitutes the base of an exponent?

General Tip: Encourage students to *think* about syntax before inputting an expression into a calculator.

- $(-1)^2$ often gets inputted incorrectly as -1^2 .
- $(\frac{2}{3})^2$ can get inputted incorrectly as $2/3^2$.

8.2 Exponent Properties Involving Quotients

Learning Objectives

At the end of this lesson, students will be able to:

- Use the quotient of powers property.
- Use the power of a quotient property.
- Simplify expressions involving quotient properties of exponents.

Vocabulary

Terms introduced in this lesson:

quotient rule for exponents
power rule for quotients

Teaching Strategies and Tips

Allow students to infer in Example 1 that *canceling* like factors is equivalent to finding the difference in the exponents of the factors.

- Therefore, the quotient rule for exponents is based on *canceling* like factors.

Additional Example:

Simplify the expression using the quotient rule.

$$\frac{x^4}{x^3}$$

Solution:

$$\frac{x^4}{x^3} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = \frac{x}{1} = x.$$

Notice also that:

$$\frac{x^4}{x^3} = x^{4-3} = x^1 = x \text{ by the quotient rule.}$$

Use Example 2 to show what happens when a larger exponent is subtracted from a smaller exponent.

- Encourage students to do the division longhand.
- Allow students to come to the conclusion that positive exponents result when subtracting smaller exponents from larger exponents.

Additional Example:

Simplify the expression leaving all powers positive.

$$\frac{m^3n^2}{m^4n^3}$$

Solution: Subtract the smaller exponents from the larger exponents and leave the result in the denominator (where the larger exponents were).

$$\frac{m^3n^2}{m^4n^3} = \frac{1}{m^{4-3}n^{3-2}} = \frac{1}{m^1n^1} = \frac{1}{mn}$$

Use Examples 1c, 3b, and 6 to show that the quotient rule is applied separately for each factor in the expression.

Additional Examples:

a. $\frac{2xy^2}{6x^3y}$

b. $\frac{72x^5y^2z}{144x^3y^2z^4}$

c. $\frac{56x^2y^2z^5}{18x^8yz}$

Combine the quotient and power rules in Examples 5-7.

- Suggest that students include each step in their solution.
- Remind students to follow the order of operations.

Additional Examples:

a. $\left(\frac{24m^3}{10m^4}\right)^3 \cdot \left(\frac{n^{12}}{n^5}\right)^2$

b. $\left(\frac{12x^2}{18y^3}\right)^2 \cdot \frac{x^3}{x^2}$

c. $\left(\frac{65x^5y^2}{145x^3y^2}\right)^3$

Error Troubleshooting

In *Review Question 7*, have students apply the quotient rule first. This will save students some work in simplifying because the numbers will be smaller. See also Examples 4a and 4b; and *Review Questions 11, 13, and 14*.

8.3 Zero, Negative, and Fractional Exponents

Learning Objectives

At the end of this lesson, students will be able to:

- Simplify expressions with zero exponents.
- Simplify expressions with negative exponents.
- Simplify expression with fractional exponents.
- Evaluate exponential expressions.

Vocabulary

Terms introduced in this lesson:

zero rule for exponents

negative exponents

negative power rule for exponents

fractional exponents

radical/ root

Teaching Strategies and Tips

Use this lesson to show how exponent rules apply to zero, negative, rational, and irrational powers.

Use Examples 1 and 2 to show that moving a factor over a fraction bar *changes the sign of its exponent*. Draw the analogy to moving a term over the equal sign.

Example:

$$\frac{x}{y^2} = \frac{xy^{-2}}{1}$$

Remind students that negative exponents turn an expression into its *reciprocal* with a positive exponent and have nothing to do with the overall expression's sign.

Example:

$$(5x)^{-3} \neq -\frac{1}{(5x)^3}, \text{ but } (5x)^{-3} = \frac{1}{(5x)^3} = \frac{1}{125x^3}.$$

Some students may try to incorrectly cancel the two negatives in an expression such as $(-2)^{-5}$. Point out that $(-2)^{-5} \neq (\cancel{-}2)^{-5}$, but $(-2)^{-5} = \frac{1}{(-2)^5} = -\frac{1}{32}$.

Additional Examples:

Write the following expressions without fractions.

a. $-\frac{1}{xy}$

b. $\frac{xz}{y^5}$

c. $\frac{12m}{n^2}$

Write the following expressions without negative exponents.

a. $-\frac{1}{x^{-1}y^{-1}}$

b. $\frac{x^{-3}z}{12y^{-5}}$

c. $8^{-2}a^{-2}b^3c^{-4}$

Write the following expressions without negative exponents. If an expression cannot be simplified, write “cannot be simplified”.

a. x^k

b. x^{-k}

c. $-x^k$

d. $(-x)^k$

e. $-x^{-k}$

f. $(-x)^{-k}$

In Example 3, suggest that students apply the exponent rules on each number and variable separately, which is possible by the commutative property. See also *Review Questions 10-16*.

Point out that there are two ways to begin *Review Question 9*.

- For problems that can be simplified in several ways, it may not always be clear to students which step should be first.

Work out additional examples:

Simplify the following expressions so that no negative exponents appear in the answer.

a. $(x^{-2}y^3)^{-5}$

Hint: Either use the power rule first or take care of the negative exponents first.

b. $\left(\frac{a^{-1}}{b^{-1}}\right)^{-1}$

Hint: Either simplify inside parentheses first or use the power rule.

c. $\left(\frac{x^2x^{-1}}{x^3}\right)^2$

Hint: Simplifying in the parentheses first will save a few steps.

d. $\left(x^{\frac{7}{3}}x^6\right)^3$

Hint: Apply the power rule first, thereby avoiding having to make common denominators later.

e. (Optional) $((x^2)^{-5})^3$

Hint: Either multiply the two inner exponents first or multiply the two outer exponents first.

Encourage students to ask *why* zero, negative, rational, and irrational powers are all valid exponents. If the following definitions are made: $x^0 = 1$, $x^{-n} = \frac{1}{x^n}$, and $x^{\frac{m}{n}} = \sqrt[n]{x^m}$ then the product, quotient, and power rules hold for all real

numbers.

Error Troubleshooting

In Example 5a, remind students to make common denominators.

General Tip: In more complicated expressions as in Example 7, have students write out each step line-by-line to avoid making algebraic errors.

General Tip: Remind students to write a 1 (*not* 0) after completely canceling a factor. For example, $\frac{x}{x^3} = \frac{1}{x^2}$.

8.4 Scientific Notation

Learning Objectives

At the end of this lesson, students will be able to:

- Write numbers in scientific notation.
- Evaluate expressions in scientific notation.
- Evaluate expressions in scientific notation using a graphing calculator.

Vocabulary

Terms introduced in this lesson:

**power of
unit
scientific notation
decimal point**

Teaching Strategies and Tips

The basic idea in this lesson shows students that moving a number's decimal point is equivalent to multiplying by a *power of 10*.

- The advantage is to shorten long numbers.
- Long numbers can be large or small.

Additional Examples:

Write the numbers in scientific notation.

a. 314159000

Solution: Move the decimal point left 8 places until there is only one digit to the left of it. Compensate for moving the decimal left (same as dividing) by multiplying by an equivalent number of 10s. Answer: 3.14159×10^8 .

b. 0.00000234

Solution: Move the decimal point right 6 places until there is only one digit to the left of it. Compensate for moving the decimal right (same as multiplying) by dividing by an equivalent number of 10s. Answer: 2.34×10^{-6} .

There are two steps for writing a number in scientific notation:

- Step 1: Count the number of places the decimal point is moved until there is one (non-zero) digit to the left of the new decimal point.

- Step 2: Multiply the new number by 10 to the power of the count in step 1. Make this exponent positive if the decimal point moved left, and negative if moved right.

Remind students that an integer such as 34,542,100 has an implied decimal point after the last digit 0.

Have students read out loud numbers in scientific notation. For example, read 7.45×10^8 as “7.45 times 10 to the power of 8”.

Error Troubleshooting

Remind students in Example 7 that 1% must be written as $\frac{1}{100}$.

General Tip. Remind students that the power of 10 does not apply to the number in front. For example, $6.3 \times 10^{-4} \neq \frac{1}{6.3 \times 10^4}$, but $6.3 \times 10^{-4} = \frac{6.3}{10^4}$.

8.5 Exponential Growth Functions

Learning Objectives

At the end of this lesson, students will be able to:

- Graph an exponential growth function.
- Compare graphs of exponential growth functions.
- Solve real-world problems involving exponential growth.

Vocabulary

Terms introduced in this lesson:

exponential function
exponential growth
exponentially increasing

Teaching Strategies and Tips

To help students see the differences between different functions, graph $y = 2^x$, $y = x$, and $y = x^2$ side-by-side. Use a table of values for each function.

Students may not make the connection between $y = 2^x$ and a quantity that doubles. Have students reconstruct the table in Example 1 and have them note the effect on the y -value as x is increased by 1.

Use Examples 2 and 3 to show students what happens when the constant A in the function $y = A \cdot b^x$ is changed.

- Compare graphs side-by-side for several values of A .

Use Example 4 to show students what happens when the constant b in the function $y = A \cdot b^x$ is changed.

- Compare graphs side-by-side for several values of b .

Allow students to make essential observations:

- All exponential functions have the same basic shape: start small and grow fast.
- At $x = 0$, the y -value is equal to A . Therefore, A is the y -intercept and called the initial value. Draw the analogy with b in the linear equation $y = mx + b$.
- The larger the base, the faster the y -values will increase.

Use Examples 6 and 7 to show students a general way to find the base b .

Have students put their answers back in context of the problem. For example, despite the negative, the number $x = -5$ in Example 6 is useful; the interpretation is 5 years ago .

Error Troubleshooting

In *Review Questions* 1-4, have students include 0 and negative values for x . Remind students that negative exponents mean division (reciprocals).

In *Review Question* 6, remind students to convert the percent to a decimal.

8.6 Exponential Decay Functions

Learning Objectives

At the end of this lesson, students will be able to:

- Graph an exponential decay function.
- Compare graphs of exponential decay functions.
- Solve real-world problems involving exponential decay.

Vocabulary

Terms introduced in this lesson:

exponential decay
asymptote
exponentially decreasing
half-life
depreciation

Teaching Strategies and Tips

Use the introductory problem to motivate the exponential decay model.

Suggest to students that exponential decay has the same general form as exponential growth, $y = A \cdot b^{-x}$.

- Allow students to compare and contrast them. For example, unlike exponential growth, exponential decay models have *two* forms; for instance, $y = \left(\frac{1}{5}\right)^x$ and $y = 5^{-x}$.

Students benefit from comparing graphs of exponential decay and growth functions side-by-side. Construct tables for each function.

Error Troubleshooting

Encourage students to construct tables in *Review Questions* 1-4. Suggest that they use zero and some negative numbers for x .

In *Review Question* 5, remind students to convert the percent to decimal.

8.7 Geometric Sequences and Exponential Functions

Learning Objectives

At the end of this lesson, students will be able to:

- Identify a geometric sequence.
- Graph a geometric sequence.
- Solve real-world problems involving geometric sequences.

Vocabulary

Terms introduced in this lesson:

geometric sequence
common ratio
term,
discrete, continuous

Teaching Strategies and Tips

Use this lesson to show how exponential functions and geometric sequences are related.

- In a geometric sequence, terms are found by multiplying the same constant to the previous term.
- For exponential functions, when the input variable is increased by 1, the output variable changes by the value of the growth rate.

Point out that students have two important models of growth in the real world: linear and exponential. Have students compare and contrast them.

- The terms of an arithmetic sequence are said to grow “linearly.” The terms of a geometric sequence are said to grow “exponentially.”
- The discrete model of linear growth is $S_n = a + nd$; where a is the first term and d the common difference. The continuous model is $y = mx + b$; where b is the y -intercept and m the slope. The discrete model of exponential growth is $S_n = ar^{n-1}$; where the first term is a , and r is the common ratio. The continuous model is $y = Ab^x$.

Additional Examples:

1. Find the common ratio of each geometric sequence.

- a. 3, 12, 48, 192, ...
- b. $-1, 1, -1, 1, -1, \dots$
- c. $6, -2, \frac{2}{3}, -\frac{2}{9}, \dots$
- d. $-10, -80, -640, -5120, \dots$

Tips: Have students create scatterplots of the sequences in their calculators.

- Allow them to discover how each is related to the value of the common ratio and the sign of the first term.
- Have them develop an explicit formula for each sequence.

2. A company is offering to pay you one penny on your first day at work and each day after that your salary would triple. Assume you work five days a week.

- a. Fill out the tables representing your daily salary for your first two weeks at work, assuming you take the job.

Monday	Tuesday	Wednesday	Thursday	Friday
0.01				

Monday	Tuesday	Wednesday	Thursday	Friday

- b. Write an equation to model the growth of your salary.
- c. How much will you make at the end of the third week?
3. Solve for the following.
- a. In a geometric sequence, $a_1 = 105$ and $a_5 = 8505$. Find r .
- b. In a geometric sequence, $a_7 = 4096$ and $r = 2$. Find a_1 .

Error Troubleshooting

In Example 1b and *Review Questions 5* and *6*, students run into difficulty because there are no *consecutive* terms that can be divided to find the common ratio. Have them use unknowns for the sequence terms starting with the first given term. Solve the resulting equation. See the hint at the end of Example 1b.

8.8 Problem-Solving Strategies (reprise Make a Table; Look for a Pattern)

Learning Objectives

At the end of this lesson, students will be able to:

- Read and understand given problem situations.
- Make tables and identify patterns.
- Solve real-world problems using selected strategies as part of a plan.

Vocabulary

Terms introduced in this lesson:

compound interest

population decrease

intensity (loudness) of sound

Teaching Strategies and Tips

Remind students of the four-step problem-solving plan.

In Example 1, students must find the compound interest formula on their own by looking for a pattern. Guide them through the first two years until the pattern becomes evident.

Have students check their answers to the examples presented in this lesson by finding b , the growth or decay rate, instead of looking for a pattern. Have them compute $(x\%)A + A = \left(1 + \frac{x}{100}\right)A$. The method was presented in Examples 6 and 7 in lesson *Exponential Growth Functions*.

Encourage students to construct tables and look for patterns in *Review Questions* 1-4. Suggest that they only check their work using explicit formulas.

Error Troubleshooting

In *Review Questions* 1-4, remind students to convert the percents to decimals.

CHAPTER 9 TE Factoring Polynomials

CHAPTER OUTLINE

- 9.1 ADDITION AND SUBTRACTION OF POLYNOMIALS
- 9.2 MULTIPLICATION OF POLYNOMIALS
- 9.3 SPECIAL PRODUCTS OF POLYNOMIALS
- 9.4 POLYNOMIAL EQUATIONS IN FACTORED FORM
- 9.5 FACTORING QUADRATIC EXPRESSIONS
- 9.6 FACTORING SPECIAL PRODUCTS
- 9.7 FACTORING POLYNOMIALS COMPLETELY

Overview

In this chapter, students add, subtract, multiply, and simplify polynomials. Students learn to recognize special products of binomials and then move on to factoring trinomials.

Suggested Pacing:

Addition and Subtraction of Polynomials - 1 hr Multiplication of Polynomials - 1 hr Special Products of Polynomials - 1 hr Polynomial Equations in Factored Form - 1 – 2 hrs Factoring Quadratic Expressions - 1 – 2 hrs Factoring Special Products - 1 hr Factoring Polynomials Completely - 2 – 3 hrs

If you would like access to the Solution Key FlexBook for even-numbered exercises, the Assessment FlexBook and the Assessment Answers FlexBook please contact us at teacher-requests@ck12.org.

Problem-Solving Strand for Mathematics

In this chapter, *Factoring Polynomials*, our exploration is focused on the problem-solving techniques *Make a Systematic List* and *Draw a Picture/Use a Model (Visualizing)*.

The FlexBook repeatedly illustrates systematic lists of possible factors, particularly in lessons Factoring Quadratic Expressions, Factoring Special Products, and Factoring Polynomials Completely. When working with these lessons, point out the organizational scheme the authors have used to list all possible factors; this can be enlightening and give students a model from which to work. By making a systematic list, students begin to see patterns that can help them make better “educated guesses.” An organized list will also help them find all possibilities.

The use of a systematic list applies in this unit not only to the sets of possible factors and the signs of these factors, but also to the techniques that should be applied before deciding whether or not a polynomial is prime.

Alignment with the NCTM Process Standards

Among the NCTM Process Standards, Communication, Connections, and Representation are most directly represented in this unit. By making a systematic, orderly list, students organize and consolidate their mathematical

thinking (COM.1) and communicate their thinking coherently and clearly to peers and teachers (COM.2). Through classroom discussions and sharing approaches and projects, they will analyze and evaluate the mathematical thinking and strategies of others (COM.3) and will recognize and use connections between algebraic and geometric representations (CON.1). The tree diagrams, tables, charts, and matrices that students create to organize, record, and communicate mathematical ideas (R.1) are excellent tools for displaying data systematically. In studying the illustrations provided in the FlexBook and developing their own visualizations and models, students will have the opportunity to select, apply, and translate among mathematical representations (R.2) and to use representations to model and interpret physical, social, and mathematical phenomena (R.3).

- COM.1 - Organize and consolidate their mathematical thinking through communication.
- COM.2 - Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
- COM.3 - Analyze and evaluate the mathematical thinking and strategies of others.
- CON.1 - Recognize and use connections among mathematical ideas.
- R.1 - Create and use representations to organize, record, and communicate mathematical ideas.
- R.2 - Select, apply, and translate among mathematical representations to solve problems.
- R.3 - Use representations to model and interpret physical, social, and mathematical phenomena.

9.1 Addition and Subtraction of Polynomials

Learning Objectives

At the end of this lesson, students will be able to:

- Write a polynomial expression in standard form.
- Classify polynomial expression by degree.
- Add and subtract polynomials.
- Problem-solve using addition and subtraction of polynomials.

Vocabulary

Terms introduced in this lesson:

polynomial

term

coefficient

constant

degree

cubic term, quadratic term, linear term

nth order term

monomial, binomial

standard form

leading term

leading coefficient

rearranging terms

like terms, collecting like terms

Teaching Strategies and Tips

There are a large number of new terms in this lesson. Introduce new vocabulary with concrete, specific examples. It is also helpful to provide examples of what the new word does *not* mean.

- Polynomials consist of terms with variables of nonnegative integer powers. Polynomials can have more than one variable.

Examples:

These are polynomials:

$$\begin{aligned}
 & -\sqrt{12}x^8 - x^5 + \pi \\
 & x^5 + x^4 - x^3 + x^2 - x + 1 \\
 & -\frac{3}{7} \\
 & x^2 + y^2 \\
 & xy \\
 & 2x^2 - 4xy + 1
 \end{aligned}$$

These are not polynomials:

$$\begin{aligned}
 & \sqrt{x} + x + 2 \\
 & y^{4.2} - x^{3.5} + x^{1.1} - x + 1 \\
 & x + \frac{1}{x} - 3 \\
 & 2^x + 8 \\
 & x^{-2} + x^{-1} + 1
 \end{aligned}$$

Have students explain their answers. Suggest that they use explanations such as:

This is not a polynomial because...

...it has a negative exponent. ...it has a radical. ...the power of x appears in the denominator. ...it has a fractional exponent. ...it has an exponential term.

- Terms are added or subtracted “pieces” of the polynomial.

Examples:

The polynomial $x^5 + x^4 - x^3 + x^2 - x + 1$ has 6 terms.

The polynomial $2x^2 - 4xy + 1$ has 3 terms; it is called a *trinomial*.

$x^2 + y^2$ is a *binomial* because it has 2 terms.

xy and $-\frac{3}{7}$ are 1-term polynomials and are called *monomials*.

In the polynomial $-2x^5 + 7x^3 - x + 8$, the 8 is a term; but neither $-2, 5, 7$, nor 3 are terms. $-x$ is another term; but x^3 is not. $7x^3$ is a term.

- The constant term is that number appearing by itself without a variable.

Examples:

In the polynomial $-2x^5 + 7x^3 - x + 8$, the 8 is the only constant term.

The polynomial $x^2 + y^2$ has no constant terms.

- Coefficients are numbers appearing in terms in front of the variable.

Examples:

$2x^2 - 4xy + 1$. The coefficient of the first term is 2. The coefficient of the second term is -4 .

$x^5 + x^4 - x^3 + x^2 - x + 1$. The coefficient of each of the terms is 1.

9.1. ADDITION AND SUBTRACTION OF POLYNOMIALS

- In standard form, a polynomial is arranged in decreasing order of powers; terms with higher exponents appear to the left of other terms.

Examples:

These polynomials are in standard form:

$$x^5 + x^4 - x^3 + x^2 - x + 1$$

$$xy + x - y - 1$$

These polynomials are not in standard form:

$$1 - x + x^2 - x^3 + x^4 + x^5$$

$$xy + x^2y^2 - 1$$

- The first term of a polynomial in standard form is called the leading term, and the coefficient of the leading term is called the leading coefficient.

Examples:

The leading term and leading coefficient of the polynomial $2x^2 - 4xy + 1$ are $2x^2$ and 2, respectively.

The leading term and leading coefficient of the polynomial $9x^2 + 8x^3 + x + 1$ are $8x^3$ and 8, respectively. Remind students to write polynomials in standard form.

- Like terms are terms with the same variable(s) to the same exponents. Like terms may have different coefficients. A polynomial is simplified if it has no terms that are alike.

Examples:

These are like terms:

$$2x^3 \text{ and } -8x^3$$

$$-xy \text{ and } 17.2xy$$

$$2x, -4x, \text{ and } \sqrt{2}x$$

$$-4, \pi, \text{ and } \sqrt{2}$$

These are not like terms:

$$x^2y \text{ and } xy^2$$

$$x^2y^2 \text{ and } x^2 + y^2$$

The polynomial $-x^3 + 3.1x^2 - 4x^2 + x - 2$ is not simplified.

- The degree of a term is the power (or the sum of powers) of the variable(s). The constant term has a degree of 0. The degree of a polynomial is the degree of its leading term. Encourage students to name polynomials by their degrees: cubic, quadratic, linear, constant.

Examples:

The term $-8x^3$ has degree 3.

The term $7.1x^2y^2$ has degree 4.

$x^5 + x^4 - x^3 + x^2 - x + 1$ is a fifth-degree polynomial.

$9x^2 + 8x^3 + x + 1$ is a cubic polynomial. Remind students to write polynomials in standard form.

Assess student vocabulary by asking them to determine all parts (terms, leading term, coefficients, leading coefficient, constant term) of a given polynomial and have them describe it in as many ways as they can (its degree, whether it is in standard form, number of variables, etc.) See Example 1-3.

When adding or subtracting polynomials, suggest that students do so vertically. The vertical or column format helps students keep terms organized.

Example:

Subtract and simplify.

$$4x^2 + 2x + 1 - (3x^2 + x - 4)$$

Solution: Subtract vertically. Keep like terms aligned.

$$\begin{array}{r} 4x^2 + 2x + 1 \\ - 3x^2 - x + 4 \\ \hline x^2 + x + 5 \end{array}$$

When simplifying like terms, suggest that students rearrange the terms into *groups* of like terms first. This is especially helpful in *Review Questions* 11, 12, and 16. See also Example 4.

Error Troubleshooting

General Tip: Remind students to distribute the minus sign to *every* term in the second polynomial when subtracting two polynomials. See Example 6 and *Review Questions* 13-16.

When simplifying polynomials, such as in Example 4b and *Review Questions* 12 and 16, remind students that like terms must have the same variables and exponents.

In Example 6, remind students that to *subtract from B* means $B - A$ and not the other way around.

Example:

Subtract $-2m^2 + 3n^2 + 4mn - 1$ from $-2n^2 - 7 + 2mn + 8m^2$.

Hint: Setup the problem as $-2n^2 - 7 + 2mn + 8m^2 - (-2m^2 + 3n^2 + 4mn - 1)$. Then distribute the negative inside the parentheses to every term. Group like terms.

General Tip: Some students will give the incorrect degree of a polynomial; remind students write polynomials in standard form and then look for the leading term.

General Tip: Students can check their answers by plugging in a simple value for the variable in the original polynomials and simplified polynomial and check if the results have the same value.

Example:

Subtract $-2m^2 + 3n^2 + 4mn - 1$ from $-2n^2 - 7 + 2mn + 8m^2$.

Solution:

Distribute. $-2n^2 - 7 + 2mn + 8m^2 - (-2m^2 + 3n^2 + 4mn - 1)$

Group like terms. $-2n^2 - 7 + 2mn + 8m^2 + 2m^2 - 3n^2 - 4mn + 1.$

$$(8m^2 + 2m^2) + (-2n^2 - 3n^2) + (2mn - 4mn) + (-7 + 1)$$

Answer: $10m^2 - 5n^2 - 2mn - 6$

Check.

Let $m = -1$ and $n = 1$.

Original: $-2n^2 - 7 + 2mn + 8m^2 - (-2m^2 + 3n^2 + 4mn - 1)$

$$\begin{aligned} & -2 \cdot 1^2 - 7 + 2 \cdot (-1) \cdot 1 + 8 \cdot (-1)^2 - (-2 \cdot (-1)^2 + 3 \cdot 1^2 + 4 \cdot (-1) \cdot 1 - 1) \\ & -3 - (-4) = 1 \end{aligned}$$

Simplified: $10m^2 - 5n^2 - 2mn - 6$

$$\begin{aligned} & 10 \cdot (-1)^2 - 5 \cdot 1^2 - 2 \cdot (-1) \cdot 1 - 6 \\ & 10 - 5 + 2 - 6 = 1 \end{aligned}$$

9.2 Multiplication of Polynomials

Learning Objectives

At the end of this lesson, students will be able to:

- Multiply a polynomial by a monomial.
- Multiply a polynomial by a binomial.
- Solve problems using multiplication of polynomials.

Vocabulary

NOT USED

Teaching Strategies and Tips

Use Example 1 to introduce polynomial multiplication beginning with monomials.

- Have students apply the product rule for exponents to each variable separately.
- Multiply the coefficients separately.

Use Examples 2 and 3 to show that the product of a monomial and any polynomial uses the distributive property.

- Show *why* the distributive property works by finding the area of a rectangle in two ways. See illustration preceding Example 2.
- After distributing, students use the method to multiply two monomials.

Use Examples 4 and 5 to demonstrate the product of two binomials. There are two approaches:

- *In-line* or *horizontal* multiplication. Use the distributive property as before.
- *Vertical* multiplication. Arrange like terms in columns. Draw the analogy to the method of multiplying two integers that students learned in grade school.

Error Troubleshooting

General Tip: Remind students to combine like terms after multiplying. Have them check their answer and make sure all like terms have been combined.

9.3 Special Products of Polynomials

Learning Objectives

At the end of this lesson, students will be able to:

- Find the square of a binomial.
- Find the product of binomials using sum and difference formula.
- Solve problems using special products of polynomials.

Vocabulary

Terms introduced in this lesson:

second-degree trinomial
square of a binomial, binomial square
sum and difference of terms
difference of squares

Teaching Strategies and Tips

In this lesson, students learn about special products of binomials.

- Have students learn to recognize the basic patterns.
- In classroom examples, use colors to denote the numbers playing the role of a and b in the formulas.

In the special formulas, point out that b is considered positive; the sign does *not* go with the term.

Example:

Square the binomial.

$$(x - 3)^2$$

Solution:

The minus sign tells us to use $(a - b)^2 = a^2 - 2ab + b^2$. Setting $a = x$ and $b = 3$ (and not -3),

$$x^2 - 2(x)(3) + (3)^2 = x^2 - 6x + 9$$

Error Troubleshooting

General Tip: Students commit a very common error when they write, for example, $(x + y)^2 = x^2 + y^2$; that is, they distribute the exponent over addition instead of multiplication.

- The power rule for products does not apply to sums or differences within the parentheses. In general, $(x^n + y^m)^p \neq x^{np} + y^{mp}$.
- Students can learn to avoid this mistake by recalling the exponent definition. Therefore, a polynomial raised to an exponent means that the polynomial is multiplied by itself as many times as the exponent indicates. For example:

$$(x + y)^2 = (x + y)(x + y)$$

- See *Review Questions* 1-4, especially Problem 4.

9.4 Polynomial Equations in Factored Form

Learning Objectives

At the end of this lesson, students will be able to:

- Use the zero-product property.
- Find greatest common monomial factor.
- Solve simple polynomial equations by factoring.

Vocabulary

Terms introduced in this lesson:

factoring, factoring a polynomial

expanded form

factored form

zero product property

factoring completely

common factor

greatest common monomial factor

polynomial equation

Teaching Strategies and Tips

Use the introduction to motivate factoring.

- The reverse of distribution is called factoring.
- Whereas before students were learning the direction $(a + b)(x + y) \Rightarrow ax + bx + ay + by$; they will now learn to “put it back together”: $ax + bx + ay + by \Rightarrow (a + b)(x + y)$.
- Students realize that polynomials can be expressed in expanded or factored form

Teachers may decide to have their students pull common factors out one at a time, instead of factoring the GCF in one step.

Error Troubleshooting

In *Review Questions* 9 and 12-16, remind students to set the monomial factor ($x, y, a,$ or b) equal to zero.

- Caution students against dividing by variables. In doing so, they will lose 0 as a solution. See also Example 6.

General Tip: Check that students are using the zero-product property correctly.

Examples:

a. *Solve for x.*

$$(x + 3)(x - 4) = 8$$

(Are students incorrectly setting each factor equal to 8?)

b. *Solve for x.*

$$(x + 3)(x - 4) - 2 = 0$$

(Are students incorrectly setting each factor equal to 0?)

General Tip: Remind students when factoring the GCF out of itself to leave a 1.

For example, $6ax^2 - 9ax + 3a \neq 3a(2x^2 - 3x)$; but $6ax^2 - 9ax + 3a = 3a(2x^2 - 3x + 1)$. See Example 5b and *Review Questions 3* and *15*.

General Tip: Have students check their work by expanding the factored polynomial.

- By checking a problem worked out as $6ax^2 - 9ax + 3a = 3a(2x^2 - 3x)$, students will convince themselves that a 1 is missing.

General Tip: Suggest that students look carefully over the remaining terms after having factored out the GCF so as to not leave any other common factors.

9.5 Factoring Quadratic Expressions

Learning Objectives

At the end of this lesson, students will be able to:

- Write quadratic equations in standard form.
- Factor quadratic expressions for different coefficient values.
- Factor when $a = -1$.

Vocabulary

Terms introduced in this lesson:

quadratic polynomial
quadratic trinomials

Teaching Strategies and Tips

In this lesson, students learn to factor quadratic polynomials according to the signs of a , b , and c :

- $a = 1, b > 0, c > 0$. See Examples 1-4.

Additional Examples:

Factor.

a. $x^2 + 15x + 26$. Answer: $(x + 13)(x + 2)$

b. $x^2 + 13x + 40$. Answer: $(x + 8)(x + 5)$

c. $x^2 + 20x + 75$. Answer: $(x + 15)(x + 5)$

- $a = 1, b < 0, c > 0$. See Examples 5 and 6.

Additional Examples:

Factor.

a. $x^2 - 17x + 42$. Answer: $(x - 14)(x - 3)$

b. $x^2 - 21x + 90$. Answer: $(x - 15)(x - 6)$

c. $x^2 - 14x + 48$. Answer: $(x - 6)(x - 8)$

- $a = 1, c < 0$. See Examples 7-9.

Additional Examples:

Factor.

a. $x^2 - 15x - 54$. Answer: $(x - 18)(x + 3)$

b. $x^2 + 7x - 60$. Answer: $(x + 12)(x - 5)$

c. $x^2 - 16x - 192$. Answer: $(x - 24)(x + 8)$

- $a = -1$. See Example 10.

Additional Examples:

Factor.

a. $-x^2 - 4x + 60$. Answer: $-(x - 6)(x + 10)$

b. $-x^2 + 14x - 40$. Answer: $-(x - 10)(x - 4)$

c. $-x^2 - 25x - 156$. Answer: $-(x + 12)(x + 13)$

- Allow students to infer that if $c > 0 (a = 1)$, then the factorization will be either of the form $(__ + __)(__ + __)$ or $(__ - __)(__ - __)$ (same signs). If $c < 0 (a = 1)$, then use the form $(__ - __)(__ - __)$ (different signs).
- See summary at the end of the lesson for a list of procedures and examples for each case.

Emphasize that factoring is the reverse of multiplication.

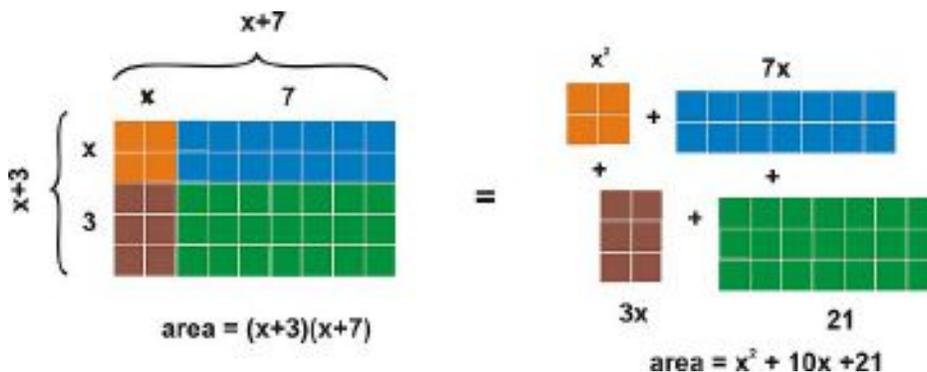
- Use an example such as $(x + 3)(x + 7) = x^2 + 10x + 21$ in which the binomials are expanded one step at a time to motivate factoring.
- Demonstrate that factoring is equivalent to putting squares and rectangles back together into larger rectangles.

Example:

Multiply.

$$(x + 3)(x + 7).$$

Solution. The diagram shows that $(x + 3)(x + 7) = x^2 + 10x + 21$. Observe that it also shows how to factor $x^2 + 10x + 21$.



Suggest that students stop listing the possible products for c after the correct choice is evident.

9.5. FACTORING QUADRATIC EXPRESSIONS

Error Troubleshooting

General Tip: For quadratic trinomials with $a = -1$, remind students to factor -1 from *every* term. Remind students to include it in their final answer.

9.6 Factoring Special Products

Learning Objectives

At the end of this lesson, students will be able to:

- Factor the difference of two squares.
- Factor perfect square trinomials.
- Solve quadratic polynomial equation by factoring.

Vocabulary

Terms introduced in this lesson:

recognizing special product
factoring perfect square trinomials
quadratic polynomial equations
double root

Teaching Strategies and Tips

Emphasize that students are reversing the special-products formulas introduced three lessons ago.

Have students use the vocabulary:

- $a^2 - b^2$ is a difference of squares.
- $(a + b)(a - b)$ is the product of a sum and difference.
- $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ are perfect square trinomials.
- $(a + b)^2$ and $(a - b)^2$ are squares of binomials.

The key to factoring special products is recognizing the special form, but also determining what a and b are.

- Recognizing perfect integer squares, for example, may be difficult to some students. Suggest that students break numbers down into prime factorization first. See Example 2.

Remind students to pull out -1 and/or the GCF in a polynomial before attempting to factor it. This simplifies the task dramatically.

Error Troubleshooting

General Tip: Remind students to check their solutions by substituting each in the original equation.

Review Question 8 is quadratic-like. Show students that $x^4 = (x^2)^2$.

9.7 Factoring Polynomials Completely

Learning Objectives

At the end of this lesson, students will be able to:

- Factor out a common binomial.
- Factor by grouping.
- Factor a quadratic trinomial where $a \neq 1$.
- Solve real-world problems using polynomial equation.

Vocabulary

Terms introduced in this lesson:

factoring strategy
factor completely
recognize special products
common monomial
factor by grouping

Teaching Strategies and Tips

Have students learn the four strategies for completely factoring a polynomial provided in the introduction.

Use Examples 3-5 to show a new factoring technique called “factoring by grouping.”

Knowing when a polynomial is completely factored may be difficult to some students. Suggest that they check each factor to see if it can be factored any more. For example, a polynomial such as $(x^2 - 4)(x^2 + 1)$ is not completely factored.

Factoring is easier when the leading term has a positive coefficient. Suggest that students include the negative as part of the GCF right in the beginning.

Error Troubleshooting

NONE

CHAPTER 10 TE Quadratic Equations and Quadratic Functions

CHAPTER OUTLINE

10.1 GRAPHS OF QUADRATIC FUNCTIONS

10.2 QUADRATIC EQUATIONS BY GRAPHING

10.3 QUADRATIC EQUATIONS BY SQUARE ROOTS

10.4 QUADRATIC EQUATIONS BY COMPLETING THE SQUARE

10.5 QUADRATIC EQUATIONS BY THE QUADRATIC FORMULA

10.6 THE DISCRIMINANT

10.7 LINEAR, EXPONENTIAL AND QUADRATIC MODELS

10.8 PROBLEM SOLVING STRATEGIES: CHOOSE A FUNCTION MODEL

Overview

This chapter introduces students to quadratic functions and their graphs. Students solve quadratic equations by completing the square and using the quadratic formula. They learn that the discriminant determines the number of roots of a quadratic equation. Students identify linear, exponential, and quadratic equations and finally learn to choose a model.

Suggested pacing:

Graphs of Quadratic Functions - 1 hr Quadratic Equations by Graphing - 1 hr Quadratic Equations by Square Roots - 0.5 hrs Quadratic Equations by Completing the Square - 1 hr Quadratic Equations by the Quadratic Formula - 1 hr The Discriminant - 0.5 hr Linear, Exponential and Quadratic Models - 1 – 2 hrs Problem Solving Strategies: *Choose a Function Model* - 1 – 2 hrs

If you would like access to the Solution Key FlexBook for even-numbered exercises, the Assessment FlexBook and the Assessment Answers FlexBook please contact us at teacher-requests@ck12.org.

Problem-Solving Strand for Mathematics

In the earlier parts of this unit, *Quadratic Equations and Quadratic Functions*, a great deal of emphasis is placed on observing patterns that emerge when various quadratics are graphed. The classic parabolic curve, as viewed in example one of the opening lesson, alters in its width, orientation (opening up or down), and/or position on the axis in subsequent examples according to discernible differences in its equation in standard form. And yet, very importantly, it is noted that the shape of a parabola always remains symmetric.

From the beginning of the unit, the text urges students to look for patterns, compare alternative approaches, and make an informed choice between these approaches in order to solve problems efficiently. For us as teachers, taking time to examine connections between various methods and investigate both their visual interpretations and future applications is well worth class-time discussion and focus. Initially students should be encouraged to compare alternative

approaches to solving quadratic equations for their relative simplicity and efficiency. Secondly, students should be urged to make explicit connections between methods and the visualization each method supports. Here again we apply the adage ascribed to Polya: “Better to solve one problem five ways than to solve five different problems.” Thirdly, students should learn that a method such as completing the square, though perhaps less mechanically simple than applying the quadratic formula, is worth learning well because it is helpful for rewriting the equations of circles, ellipses, and hyperbolas.

In the formal problem-solving lesson of the chapter, *Choose a Function Model*, students encounter mathematical modeling, a process that begins with real-world situations and arrives at quantitative solutions through as many refinements and adjustments as is practicable to reach acceptable results. The Review Questions offer yet another opportunity for students to work in small groups. Small group, collaborative work allows students to talk about what they do understand. If a group is confused, you might suggest that the students develop two or three clarifying questions that they could present to their classmates for discussion or clarification.

Alignment with the NCTM Process Standards

This chapter, with its extensive study of quadratic equations and functions, touches on each of the process standards of Connections, Problem Solving, and Representation. Students recognize and see connections among mathematical ideas (CON.1) and understand how mathematical ideas interconnect and build on one another to produce a coherent whole (CON.2). This is particularly evident in the development of various ways to solve quadratics, finally leading to completing the square and presenting the derivation of the quadratic formula. Furthermore, within each lesson and especially in the Problem Solving Strategies lesson at the end of the chapter, real-life applications are highlighted, applying mathematics in contexts outside of mathematics itself (CON.3).

The unit is also replete with problem-solving considerations; it builds new mathematical knowledge (PS.1), solves problems that arise in mathematics and in other contexts (PS.2), and applies and adapts a variety of appropriate strategies to solve problems (PS.3). Especially in the lesson that focuses on mathematical modeling, the unit monitors and reflects on the process of mathematical problem solving (PS.4), encouraging revisions and reformulations to be certain that the real-world situation is actualized reasonably well. Much of the unit creates and uses representations (tables, graphs, equations) to organize, record, and communicate mathematical ideas (R.1), selects, applies and translates among mathematical representations to solve problems (R.2), and uses representations to model and interpret physical, social, and mathematical phenomena (R.3).

- CON.1 - Recognize and use connections among mathematical ideas.
- CON.2 - Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
- CON.3 - Recognize and apply mathematics in contexts outside of mathematics.
- PS.1 - Build new mathematical knowledge through problem solving.
- PS.2 - Solve problems that arise in mathematics and in other contexts.
- PS.3 - Apply and adapt a variety of appropriate strategies to solve problems.
- PS.4 - Monitor and reflect on the process of mathematical problem solving.
- R.1 - Create and use representations to organize, record, and communicate mathematical ideas.
- R.2 - Select, apply, and translate among mathematical representations to solve problems.
- R.3 - Use representations to model and interpret physical, social, and mathematical phenomena.

10.1 Graphs of Quadratic Functions

Learning Objectives

At the end of this lesson, students will be able to:

- Graph quadratic functions.
- Compare graphs of quadratic functions.
- Graph quadratic functions in intercept form.
- Analyze graphs of real-world quadratic functions.

Vocabulary

Terms introduced in this lesson:

quadratic equations

quadratic functions

parabola

smooth curve

general form, standard form of a quadratic function

coefficients

orientation

dilation

vertical, horizontal shift

vertex of a parabola

symmetric, line of symmetry

quadratic expression

intercept form

Teaching Strategies and Tips

Point out that students frequently come into contact with parabolas.

- The path of any projectile is part of a parabola.
- Cross-sections of satellite dishes and flashlight mirrors are parabolas.
- See introduction for more examples.

Draw several parabolas and lead students in discovering the essential features:

- Symmetry

- A maximum of two x -intercepts
- Vertices
- None of these features are shared by the other graphs students have been studying.

Use a table of values to show *why* the parabola has a U-shape.

- Allow students to observe the effect of squaring on negative numbers.
- Use a basic quadratic function such as $y = x^2$.

When graphing quadratic functions, suggest that:

- Sketches do not need to be perfect, but curves should be U-shaped rather than V-shaped. In general, join points with a smooth curve rather than with segments.
- Plot more points until the familiar curve is in view. More points are necessary to draw a parabola accurately than to graph lines, since a parabola is curved. See Example 1c.
- Knowing the vertex cuts down on the number of points to be plotted, because that is where the parabola opens. Advise students to choose at least one point on either side of the vertex.
- Not all points in a table must be plotted. This is especially true for points with large y -values which can make a y -axis too big. See Examples 1a and 1c.

Use Example 2 to introduce *intercept form*.

Additional Examples:

Find the x -intercepts and vertices of the following quadratic functions.

a. $y = x^2 + 3x - 10$

Solution: Write the quadratic function in intercept form by factoring the right side.

$$y = x^2 + 3x - 10 = (x + 5)(x - 2)$$

Set $y = 0$. So, the x -intercepts are $(-5, 0)$ and $(2, 0)$. The x -value of the vertex is halfway between the two x -intercepts:

$$\frac{-5 + 2}{2} = -1.5$$

To the y -value, plug in the x -value into the original equation:

$$y = (-1.5)^2 + 3(-1.5) - 10 = -12.25$$

The vertex is $(-1.5, -12.25)$.

b. $y = -2x^2 + 56x + 120$

Answer: The x -intercepts are $(-2, 0)$ and $(30, 0)$ and vertex is $(14, 512)$.

c. $y = -x^2 + 15x - 36$

Answer: The x -intercepts are $(12, 0)$ and $(30, 0)$ and vertex is $(7.5, 20.25)$.

Point out that the vertex lies on the line of symmetry; therefore, x -intercepts are reflections of each other and the vertex is halfway between them.

By analyzing the effects of the coefficients a and c in $y = ax^2 + c$, teachers can introduce the basic transformations: shift, stretch, and flip. Ask:

10.1. GRAPHS OF QUADRATIC FUNCTIONS

- What makes the graph of the parabola open up? Down?
- What makes a parabola wider? Narrower?
- How can the vertex of one parabola be placed higher than another?

Use graphing calculators to:

- Find intercepts, vertices, and points of intersection to any degree of precision.
- Compare several quadratic functions simultaneously.

Error Troubleshooting

NONE

10.2 Quadratic Equations by Graphing

Learning Objectives

At the end of this lesson, students will be able to:

- Identify the number of solutions of quadratic equations.
- Solve quadratic equations by graphing.
- Find or approximate zeros of quadratic functions.
- Analyze quadratic functions using a graphing calculator.
- Solve real-world problems by graphing quadratic functions.

Vocabulary

Terms introduced in this lesson:

roots

zeros

distinct solutions

double root

no real solutions

Teaching Strategies and Tips

Emphasize the connection between algebra and geometry:

- Finding the roots of a quadratic function (algebra) is equivalent to finding the x -intercepts of a parabola (geometry).
- Have students use correct vocabulary: equations have *roots* or *zeros*; graphs have *x -intercepts*.

Use Example 4 to explore the graph of an equation using a graphing calculator.

- In graph mode, use the cursor to scroll over the x -intercepts or vertex to find an approximate value of each. Approximations can be improved by zooming in.
- In graph mode, from the CALC menu, use ZERO and MAXIMUM (MINIMUM) to find an x -intercept and vertex, respectively.
- Use the built-in table to find an x -intercept (look for the row with $y = 0$) or a vertex (scroll through the table values until the y values change from increasing to decreasing; or vice-versa). By changing the table's step size, the approximation can be made better.
- Emphasize approximating roots of an equation by reading a graph is an essential skill; real-world equations rarely factor over the integers.

In Example 5, encourage students to explore the equation on a graphing calculator using the WINDOW menu. By changing parameters XMIN, XMAX, YMIN, and YMAX, students learn to find an appropriate display for any graph.

Have students interpret their answers. In Example 5, the two roots indicate the two times when the arrow is on the ground.

General Tip: In *Review Questions* 1-12, have students check their answers with a graphing calculator for added practice.

Error Troubleshooting

In *Review Question* 6, remind those students inputting the equation into a calculator to use proper syntax. The following are equivalent:

- $y_1 = (1/2)x^2 - 2x + 3$
- $y_1 = x^2/2 - 2x + 3$
- $y_1 = (x^2)/2 - 2x + 3$

10.3 Quadratic Equations by Square Roots

Learning Objectives

At the end of this lesson, students will be able to:

- Solve quadratic equations involving perfect squares.
- Approximate solutions of quadratic equations.
- Solve real-world problems using quadratic functions and square roots.

Vocabulary

Terms introduced in this lesson:

quadratic equations involving perfect squares
positive, negative square root
approximate solutions
free fall

Teaching Strategies and Tips

Remind students that:

- $\sqrt{x^2} = |x|$ and is called the principal square root.
- $|x| = k$ implies $x = \pm k$.
- Conclude that $x^2 = c$ has two solutions when $c > 0$. See Examples 1-3.
- Point out that square roots of negative numbers are not real numbers. Therefore, $x^2 = c$ has no solutions when $c < 0$.

Specify in advance the number of decimal places required of students in Examples 6 and 7 and *Review Questions* 18-22.

Error Troubleshooting

In Example 2, show students that $\sqrt{\frac{16}{9}} = \frac{4}{3}$ and $\sqrt{\frac{1}{81}} = \frac{1}{9}$. More square-roots of fractions can be found in *Review Questions* 4-6 and 8.

General Tip: Remind students after square-rooting both sides of an equation to include the \pm .

10.4 Quadratic Equations by Completing the Square

Learning Objectives

At the end of this lesson, students will be able to:

- Complete the square of a quadratic expression.
- Solve quadratic equations by completing the square.
- Solve quadratic equations in standard form.
- Graph quadratic equations in vertex form.
- Solve real-world problems using functions by completing the square.

Vocabulary

Terms introduced in this lesson:

completing the square
perfect square trinomial
quadratic equations in standard form
quadratic equations in vertex form
parabola turns up, turns down

Teaching Strategies and Tips

Use Example 1 to introduce completing the square.

- Remind students of binomial expansions to help them understand how a constant term turns the expression into a perfect square trinomial:

$$(x \pm a)^2 = x^2 \pm 2ax + a^2$$

- Point out that the leading coefficient is 1.
- In Example 3, $a \neq 1$. Students learn to factor a from the whole expression before completing the square, whether or not the terms are multiples of a . Complete the square of the resulting expression in parentheses.

Give completing the square geometrical meaning.

- Use squares and rectangles for each term of the expression.
- See paragraph preceding Example 4.
- Have students make up a quadratic expression and ask them to complete the square in the geometrical interpretation as an assignment.

There are several reasons to have students learn completing the square:

- It is used to derive the quadratic formula.
- Quadratic equations can be rewritten in vertex form.
- Equations of circles can be rewritten in graphing form.
- Necessary in calculus.

Emphasize that completing the square finds roots

- Regardless of whether the roots are integers, rational or irrational numbers.
- Without having to list all the cases, unlike factoring.

Example:

Solve the following quadratic equation:

$$2x^2 + x - 3 = 0$$

Solution: Although $2x^2 + x - 3 = (2x + 3)(x - 1)$, and therefore $x = -\frac{3}{2}$ and $x = 1$; we show that completing the square results in the same solutions.

Rewrite:

$$2x^2 + x = 3$$

Divide by the leading coefficient; this results in an equation with $a = 1$.

$$x^2 + \left(\frac{1}{2}\right)x = \frac{3}{2}$$

Add the constant to both sides of the equation:

$$x^2 + \left(\frac{1}{2}\right)x + \left(\frac{1}{4}\right)^2 = \frac{3}{2} + \left(\frac{1}{4}\right)^2$$

Factor the perfect square trinomial and simplify.

$$\left(x + \frac{1}{4}\right)^2 = \frac{3}{2} + \frac{1}{16}$$

$$\left(x + \frac{1}{4}\right)^2 = \frac{24}{16} + \frac{1}{16}$$

$$\left(x + \frac{1}{4}\right)^2 = \frac{25}{16}$$

Take the square root of both sides:

$$\begin{aligned}\sqrt{\left(x + \frac{1}{4}\right)^2} &= \sqrt{\frac{25}{16}} \\ x + \frac{1}{4} &= \pm \frac{5}{4} \\ x &= -\frac{1}{4} \pm \frac{5}{4} \\ x &= \frac{-1 \pm 5}{4} \\ \Rightarrow x &= \frac{-1 + 5}{4} = 1 \\ \Rightarrow x &= \frac{-1 - 5}{4} = -\frac{3}{2}\end{aligned}$$

Answer: $x = -\frac{3}{2}$ and $x = 1$ as expected.

Use Examples 7-9 to show how completing the square helps in graphing quadratic functions.

Error Troubleshooting

Dividing by the leading coefficient in *Review Questions* 7, 8, and 14 results in a fractional b coefficient. Remind students that dividing b by 2 is equivalent to multiplying b by $1/2$.

Additional Example:

Solve the quadratic equation by completing the square:

$$3x^2 + 5x - 3 = 0$$

Hint: Rewrite the equation:

$$3x^2 + 5x = 3.$$

And divide by the leading coefficient:

$$x^2 + \frac{5}{3}x = 1$$

To find the constant that must be added to both sides of the equation, *multiply* $5/3$ by $1/2$ and then square:

$$\left(\frac{5}{3} \cdot \frac{1}{2}\right)^2 = \left(\frac{5}{6}\right)^2$$

General Tip: Remind students to rewrite the vertex form of a quadratic equation with minus signs to find h and k correctly. See Examples 7-9.

Additional Example:

Find the vertex of the parabola with equation:

$$y + 3 = -2(x - 5)^2$$

Solution: Rewrite the equation with minus signs:

$$y - (-3) = -2(x - 5)^2$$

The vertex is at (h, k) , where $h = -3$ and $k = 5$.

General Tip: Remind students that the leading coefficient must be 1 before completing the square. They will need to divide or factor to accomplish this.

General Tip: Suggest that students rewrite equations in standard form before completing the square.

In Examples 10 and 11 and *Review Questions 25* and *26*, have students round in the *last* step.

10.5 Quadratic Equations by the Quadratic Formula

Learning Objectives

At the end of this lesson, students will be able to:

- Solve quadratic equations using the quadratic formula.
- Identify and choose methods for solving quadratic equations.
- Solve real-world problems using functions by completing the square.

Vocabulary

Terms introduced in this lesson:

quadratic formula
roots, solutions to a quadratic equation

Teaching Strategies and Tips

Emphasize that the quadratic formula comes from *completing the square* of a general quadratic equation.

- Point out that half the coefficient of x squared can be found by *multiplying* by $1/2$:

$$\left(\frac{b}{a} \cdot \frac{1}{2}\right)^2 = \left(\frac{b}{2a}\right)^2$$

- Remind students to find common denominators before simplifying the right side of the equation.

Have students rewrite each quadratic equation in standard form first. See Example 3.

Use Example 4 to show students how one goes about choosing a solving method.

Teachers are encouraged to use quadratic equations with non-integer coefficients—decimals and fractions—as additional practice problems.

Error Troubleshooting

General Tip: Some students may use the values of a , b , and c based upon the order in which the terms were given. Students will assume that a is the first coefficient, b the second, and c the last.

- Remind students rewrite each quadratic equation in standard form first. See Example 3.

General Tip: Have students simplify one step at a time when using the quadratic formula:

- Substitute.
- Simplify under the radical.
- Determine the square root.
- Simplify the numerator.
- Simplify the denominator.
- Divide.

General Tip: Some common mistakes associated with the quadratic formula are:

- Not using the minus sign that goes with a coefficient.

Example: $3x^2 + 7x - 4 = 0$. Use $a = 3$, $b = 7$, and $c = -4$. ($c \neq 4$)

- Losing the minus sign on $-b$.

Example: $3x^2 - 7x + 4 = 0$. Use $a = 3$, $b = -7$, and $c = 4$.

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(4)}}{2(3)} \text{ and not } x = \frac{-7 \pm \sqrt{(-7)^2 - 4(3)(4)}}{2(3)}$$

- Canceling a factor of the denominator with $-b$ only. Remind students that in order to cancel a factor from the denominator, it must be a common factor in every term in the numerator. Unless that factor comes out of the radical, then the factor in the denominator *cannot* be canceled.

Example: For $a = 3$, $b = 6$, and $c = -1$, $x = \frac{-6 \pm \sqrt{6^2 - 4(3)(-1)}}{2(3)} = \frac{-6 \pm \sqrt{48}}{6} \Rightarrow x = \frac{-\cancel{6} \pm \sqrt{48}}{\cancel{6}}$.

- Rounding at steps other than the last step.

10.6 The Discriminant

Learning Objectives

At the end of this lesson, students will be able to:

- Find the discriminant of a quadratic equation.
- Interpret the discriminant of a quadratic equation.
- Solve real-world problems using quadratic functions and interpreting the discriminant.

Vocabulary

Terms introduced in this lesson:

discriminant

double root

nature of solutions

distinct real solutions

non-real solutions

rational number solutions

irrational number solutions

Teaching Strategies and Tips

In this lesson, students learn that each quadratic equation is associated with a number whose sign tells how many solutions there are to that equation.

- Emphasize that the equation does *not* have to be solved.

Point out that if the discriminant is not a perfect square, then the solutions are irrational numbers. See Example 3.

Use Examples 5-7 to illustrate how useful interpreting the discriminant in context of the problem can be.

Error Troubleshooting

General Tip: Remind students to use care when substituting values into the discriminant.

10.7 Linear, Exponential and Quadratic Models

Learning Objectives

At the end of this lesson, students will be able to:

- Identify functions using differences and ratios.
- Write equations for functions.
- Perform exponential and quadratic regressions with a graphing calculator.
- Solve real-world problems by comparing function model.

Vocabulary

Terms introduced in this lesson:

ratio of the differences

constant ratio of differences

exponential, quadratic regression

Teaching Strategies and Tips

Emphasize that students should always begin a regression by drawing the scatter plot of the data. This allows them to determine which function of best fit is most appropriate, if any.

Point out:

- A regression is a curve that comes as close as possible to all the data points without being another type of function.
- Regressions do not necessarily pass through data points. Therefore, y -values will be different from those in the data set.

Remind students to check in general the differences in x -values that they are constant.

Error Troubleshooting

In *Review Questions* 4-6 and 12, remind students to calculate the differences of the differences to see if a quadratic trend exists. Students often calculate only the first set of differences.

10.8 Problem Solving Strategies: Choose a Function Model

Learning Objectives

At the end of this lesson, students will be able to:

- Read and understand given problem situations.
- Develop and use the strategy: *Choose a Function Model*.
- Develop and use the strategy: *Make a Model*.
- Plan and compare alternative approaches to solving problems.
- Solve real-world problems using selected strategies as part of a plan.

Vocabulary

Terms introduced in this lesson:

logistic regression

Teaching Strategies and Tips

The lesson presents several mathematical models arising from real-world applications in the examples.

Encourage students to read and understand the modeling process presented as a flow chart in the introduction.

In each of the examples, present and compare several regressions side-by-side. Allow students to see how accurate each is.

Error Troubleshooting

General Tip: Students rush to calculate the regression not stopping to consider its appropriateness. A scatter plot should always be consulted first.

General Tip: Have students check the data points they have inputted into a calculator twice.

General Tip: On the TI graphing calculators, students should be using $LinReg(ax + b)$ or $LinReg(a + bx)$ to perform linear regressions and not $LnReg$.

CHAPTER 11 TE Algebra and Geometry Connections; Working with Data

CHAPTER OUTLINE

- 11.1 GRAPHS OF SQUARE ROOT FUNCTIONS
 - 11.2 RADICAL EXPRESSIONS
 - 11.3 RADICAL EQUATIONS
 - 11.4 THE PYTHAGOREAN THEOREM AND ITS CONVERSE
 - 11.5 DISTANCE AND MIDPOINT FORMULAS
 - 11.6 MEASURES OF CENTRAL TENDENCY AND DISPERSION
 - 11.7 STEM-AND-LEAF PLOTS AND HISTOGRAMS
 - 11.8 BOX-AND-WHISKER PLOTS
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Overview

In this chapter, students are introduced to the square root function and learn how shifts, flips, and stretches change the shape of the graph. Students use the properties of radicals to solve radical equations and problems involving the Pythagorean theorem, distance and midpoint formulas. The chapter ends with an introduction to descriptive statistics and graphical displays.

Suggested pacing:

Graphs of Square Root Functions - 1 – 2 hrs Radical Expressions - 2 – 4 hrs Radical Equations - 1 hr The Pythagorean Theorem and Its Converse - 1 hr Distance and Midpoint Formulas - 1 hr Measures of Central Tendency and Dispersion - 1 hr Stem-and-Leaf Plots and Histograms - 1 hr Box-and-Whisker Plots - 1 hr

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Problem-Solving Strand for Mathematics

Along with the four-step problem-solving plan presented in the text, students are prompted to reflect back on strategies, evaluate their effectiveness and analyze their usefulness for future problems. This chapter, *Algebra and Geometry Connections; Working with Data*, continues to promote such an analysis.

Recognizing patterns, drawing connections, and considering extraneous solutions are all featured. When dealing with extraneous solutions, point out to students where in the procedures the potential for a false solution has been introduced. This awareness can prepare them for more sophisticated mathematics ahead.

Encourage students to think about everyday applications of the Pythagorean Theorem (students might interview carpenters, plumbers, architects, and/or structural engineers, for example) and emphasize that the distance and midpoint formulas are extensions of the basic Pythagorean theorem and “taking an average” respectively. Similarly,

with Stem-and-Leaf Plots, Histograms, and Box-and-Whisker Plots, be sure students understand the focus and strengths of each style of graph. The point with each of these tools is to represent and/or be able to interpret data in ways that make sense and communicate clearly.

Alignment with the NCTM Process Standards

This chapter, focused on algebra and geometry connections and working with data, aligns with many of the NCTM process standards. Throughout students are asked to recognize and use connections among mathematical ideas (CON.1), to appreciate how mathematical ideas interconnect and build on one another to produce a coherent whole (CON.2) and to recognize and apply mathematics in contexts outside of mathematics (CON.3). When dealing with radical equations and extraneous solutions, students apply and adapt a variety of appropriate strategies to solve problems (PS.3) and monitor and reflect on the process of mathematical problem solving (PS.4). In working with the Pythagorean theorem and its converse, students use representations to model and interpret physical and mathematical phenomena (R.3). In applying the distance and midpoint formulas students select, apply, and translate among mathematical representations to solve problems (R.2).

In the lessons which produce and analyze data communication tools, students create and use representation to organize, record, and communicate mathematical ideas (R.1); communicate their mathematical thinking coherently and clearly to peers, teachers, and others (COM.2); analyze and evaluate the mathematical thinking and strategies of others (COM. 3); and use the language of mathematics to express mathematical ideas precisely (COM.4).

- COM.2 - Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
- COM.3 - Analyze and evaluate the mathematical thinking and strategies of others.
- COM.4 - Use the language of mathematics to express mathematical ideas precisely.
- CON.1 - Recognize and use connections among mathematical ideas.
- CON.2 - Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
- CON.3 - Recognize and apply mathematics in contexts outside of mathematics.
- PS.3 - Apply and adapt a variety of appropriate strategies to solve problems.
- PS.4 - Monitor and reflect on the process of mathematical problem solving.
- R.1 - Create and use representations to organize, record and communicate mathematical ideas.
- R.2 - Select, apply, and translate among mathematical representations to solve problems.
- R.3 - Use representations to model and interpret physical, social, and mathematical phenomena.

11.1 Graphs of Square Root Functions

Learning Objectives

At the end of this lesson, students will be able to:

- Graph and compare square root functions.
- Shift graphs of square root functions.
- Graph square root functions using a graphing calculator.
- Solve real-world problems using square root functions.

Vocabulary

Terms introduced in this lesson:

increases, decreases

flip

shift

stretch

transform

Teaching Strategies and Tips

Use Example 1 to introduce the basic shape of the square root function.

Use the tables in the examples to show students the square root function's behavior numerically:

- Why it is undefined on some intervals.
- That it is everywhere increasing.
- That it rises relatively slowly.
- That the square root of a fraction is greater than the fraction. Numbers in the interval $(0, 1)$ are smaller than their square roots; $x < \sqrt{x}$ for x in the interval $(0, 1)$ and $x > \sqrt{x}$ for x in the interval $(1, \infty)$.

Use the graphs in the examples to make the following observations:

- The square root graph is half a parabola lying sideways.

Emphasize finding the domain of the square root function before making a table.

- When the expression under the square root is negative, table values will be undefined; and the graph corresponding to the interval will be empty.

Use Examples 2-10 to motivate transformations:

- Shifts, stretches, and flips allow graphing *without* constructing a table of values.
- Teachers are encouraged to use several examples to illustrate the effect of each constant on the graph.

Additional Examples:

Graph the following functions using transformations of the basic graph $y = \sqrt{x}$.

a. $y = -\sqrt{x}$

Hint: Flip about the x -axis.

b. $y = \sqrt{-x}$

Hint: Flip about the y -axis.

Remark: Student often claim that the whole function is undefined because of the negative under the radical. Point out that the domain “flips” to negative numbers.

c. $y = 2\sqrt{x}$

Hint: Stretch in the vertical direction by a factor of 2: y -values are multiplied by 2.

d. $y = \sqrt{2x}$

Hint: Point out that $y = \sqrt{2x} = \sqrt{2}\sqrt{x}$.

e. $y = \sqrt{x} + 2$

Hint: Shift the graph up by 2: y -values are increased by 2.

f. $y = \sqrt{x+2}$

Hint: Shift the graph left by 2: x -values are decreased by 2.

g. $y = -\sqrt{x} + 1$

Hint: Point out that the transformations are in the same direction. y -values are reflected across the x -axis (parallel to the y -axis) and then shifted vertically (parallel to the y -axis), in that order. Therefore, the correct sequence is to flip and then shift.

h. $y = \sqrt{-x+1}$

Hint: Have students consider what happens to an input x and do the transformations in the opposite order. Therefore, shift left, then reflect across the y -axis.

Additional Example:

Graph the following function using shifts, flips, and stretches.

a. $y = 4 + 2\sqrt{2-x}$

Solution: View $y = 4 + 2\sqrt{2-x}$ as a combination of transformations of the basic square root graph $y = \sqrt{x}$.

Start with the simpler equation: $y = \sqrt{2-x}$. If we follow an input x , then it first gets multiplied by -1 and then increased by 2. How do the graphs of $y = \sqrt{x}$ and $y = \sqrt{2-x}$ compare? $y = \sqrt{2-x}$ is a shift of $y = \sqrt{x}$ two units LEFT and then flipped across the y -axis. Note that the transformations happen in the opposite order in which an input x gets operated on.

To graph $y = 2\sqrt{2-x}$ we multiply the y -values of $y = \sqrt{2-x}$ by 2 to obtain a vertically stretched curve. Finally, to obtain the graph of $y = 4 + 2\sqrt{2-x}$, shift the graph of $y = 2\sqrt{2-x}$ four units vertically.

Encourage students to keep a list of functions they have studied so far. Include a few examples of each and their graphs. For example:

- Linear: $f(x) = mx + b$

Examples: $f(x) = x$, $f(x) = -x$, $f(x) = x + 1$, $f(x) = 2$

- Exponential: $f(x) = a \cdot b^x$

Examples: $f(x) = 2^x$, $f(x) = 2^{-x}$, $f(x) = -2^x$

- Quadratic: $f(x) = ax^2 + bx + c$

Examples: $f(x) = x^2$, $f(x) = -x^2$, $f(x) = x^2 + 1$

- Square root: $f(x) = a\sqrt{bx+c} + d$

Examples: $f(x) = \sqrt{x}$, $f(x) = \sqrt{x+1}$, $f(x) = \sqrt{x} + 1$, $f(x) = -\sqrt{x}$

Error Troubleshooting

General Tip: Students may not recognize $y = \sqrt{-x}$ as a valid function at first, stating that the square root of a negative is undefined. Explain that the function's domain is defined.

General Tip: Have students find the domain of the function they are graphing on a calculator first. This will help find an appropriate window for the graph.

In Example 12 and *Review Questions* 14-18, state beforehand the number of decimal places required of students when rounding.

11.2 Radical Expressions

Learning Objectives

At the end of this lesson, students will be able to:

- Use the product and quotient properties of radicals.
- Rationalize the denominator.
- Add and subtract radical expressions.
- Multiply radical expressions.
- Solve real-world problems using square root functions.

Vocabulary

Terms introduced in this lesson:

radical sign

even roots, odd roots

simplest radical form

rationalizing the denominator

Teaching Strategies and Tips

In this lesson, students learn that:

- Radicals reverse the operation of exponentiation.
- The index determines the kind of root.
- The square root is the only index which is not explicitly written but understood.

Use Example 1 to point out that even and odd indices handle negatives differently.

Additional Examples:

a. $\sqrt{-64}$ is not a real number.

b. $\sqrt[3]{-64}$ is defined and evaluates as -4 .

Use Example 2 to motivate rational exponents.

- One way to justify that $\sqrt[n]{a} = a^{1/n}$ is to use remind students of the power rule: $(x^m)^n = x^{mn}$.
- Let $m = \frac{1}{2}$ and $n = 2$: $(x^{\frac{1}{2}})^2 = x^1 = x$.
- Therefore, $x^{1/2}$ is a number that when squared equals x . Therefore, $x^{1/2} = \sqrt{x}$.

- A similar argument holds in general for any index: $\sqrt[n]{a} = a^{1/n}$.
- Using the power rule again: $a^{m/n} = a^{m \cdot \frac{1}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$.
- Therefore, $\sqrt[n]{a^n} = a^{n/n}$.

Have students state the radical properties in words. This can help students learn the rules:

- The product rule for radicals: The square root of the product is the product of the square roots.
- The quotient rule for radicals: The square root of the quotient is the quotient of the square roots.

Rationalizing the denominator:

- Remind students to multiply the numerator *and* denominator by the radical expression. “What you do to the top you do to the bottom.”
- Point out that rationalizing the denominator is essentially multiplying by 1; therefore, the value of the original rational expression does not change.
- Have students seek a radical expression that when multiplied with the denominator results in a perfect power.
- In the case when the denominator contains two terms, one being a radical, a good choice for the rationalization is an expression whose product is a difference of squares.
- See *Review Questions 26-33*.

Encourage students to leave their answers in radical form unless otherwise specified.

- Radical form is an exact answer.
- If a decimal is needed, the final radical can be rounded.

Have students simplify *all* radicals to simplest form to ensure that all possible like terms in the expression are combined.

For students having a difficult time adding and simplifying radical expressions, draw the analogy with combining like terms in variable expressions. For example, the expressions $-2x + 7x$ and $-2\sqrt{5} + 7\sqrt{5}$ are essentially the same.

Teachers are encouraged to be specific about when a radical is in *simplified form*:

- No fractions occur in the radicand.

Example: The expression $x^2y\sqrt{\frac{x}{4}}$ can be simplified to $\frac{x^{\frac{5}{2}}y}{2}$.

- No radicals are present in the denominator of a fraction.

Example: The expression $\frac{1}{\sqrt{x}}$ can be simplified to $\frac{\sqrt{x}}{x}$ or $x^{-\frac{1}{2}}$. Decide whether to include negative exponents in simplified form.

- The index of a radical and the exponents on any expressions in the radicand do not have common factors.

Example: The expression $\sqrt[6]{x^2}$ can be simplified to $\sqrt[3]{x}$.

- The exponents on any expressions in the radicand are less than the index.

Example: The expression $\sqrt[5]{x^9}$ can be simplified to $x\sqrt[5]{x^4}$.

- The resulting expression has as few radicals as possible.

Example: The expression $\sqrt{5} + \sqrt{20}$ can be simplified to $3\sqrt{5}$.

11.2. RADICAL EXPRESSIONS

Error Troubleshooting

In *Review Questions* 9-16 have students look for the highest possible perfect squares, cubes, fourth powers, etc. as indicated by the index of the radical. Suggest that they use factor trees as guides.

In *Review Questions* 14-16, have students treat the constants and variables separately.

General Tip: Remind students that when adding and subtracting radical expressions to combine only *like* radical terms (the same expression under the radical sign). This is analogous to combining *like terms* in variable expressions.

In *Review Question* 25, remind students to multiply the numbers outside the radical sign and the numbers inside the radical sign separately. Use the rule: $a\sqrt{b} \cdot c\sqrt{d} = ac\sqrt{bd}$.

11.3 Radical Equations

Learning Objectives

At the end of this lesson, students will be able to:

- Solve a radical equation.
- Solve radical equations with radicals on both sides.
- Identify extraneous solutions.
- Solve real-world problems using square root functions.

Vocabulary

Terms introduced in this lesson:

radical equation
radical expression
extraneous solutions

Teaching Strategies and Tips

Up to this point, students have been solving linear and quadratic equations. In this lesson, they now look at solving radical equations.

The following steps are used to solve radical equations:

- Isolate a radical.
- Square both sides of the equation (or use another appropriate power).
- Solve the new polynomial equation now free of radicals.
- Check answers in the original equation.

Use Example 2 to show that radical equations containing radicals of *any* index – not just square roots – can be solved.

- The steps are identical except for a change in the power that each side of the equation is raised to.

Use Example 4 to show that radical equations containing *more than one* radical expression can be solved.

- Isolate the most complicated radical expression and raise the equation to the appropriate power.
- Repeat the process until all radical signs are eliminated. In Example 4 and *Review Questions* 11-16, students must square both sides, twice.

Some radical equations can be made easier by reducing all terms by a common factor.

- This should occur at the beginning of the problem or after the step when both sides have been raised to a power.

Example:

Find the real solutions of:

$$4\sqrt{2x+1} = 12 - 8\sqrt{x}$$

Hint: Divide by 4 first.

$$\sqrt{2x+1} = 3 - 2\sqrt{x}$$

After squaring both sides and simplifying, the equation is:

$$12\sqrt{x} = 2x + 8$$

Divide again, this time by 2 before squaring both sides.

Point out that an equation such as $\sqrt{x+5} = -3$ can be readily answered as not having any real solutions.

Error Troubleshooting

In *Review Questions* 13-16, remind students to isolate a radical first. By not isolating, some radical will always remain in the equation and can even make the equation more complicated.

General Tip: Remind students to raise each *side* of the equation to the appropriate power, rather than term by term. Encourage students to use parentheses for each side.

General Tip: Have students always check their answers for extraneous solutions.

11.4 The Pythagorean Theorem and Its Converse

Learning Objectives

At the end of this lesson, students will be able to:

- Use the Pythagorean theorem.
- Use the converse of the Pythagorean theorem.
- Solve real-world problems using the Pythagorean theorem and its converse.

Vocabulary

Terms introduced in this lesson:

Pythagorean theorem
hypotenuse
legs
converse

Teaching Strategies and Tips

Students learn that the Pythagorean theorem relates the lengths of the sides of a right triangle to each other.

Students also learn that the Pythagorean theorem has a converse.

- It can be used to verify that a triangle is a right triangle.
- If it can be shown that the three sides of a triangle make the equation $a^2 + b^2 = c^2$ true, then the triangle is a right triangle.

Remind students of the following few prerequisites from geometry:

- A right triangle is one that contains a 90 degree angle.
- The side of the triangle opposite the 90 degree angle is called the hypotenuse.
- The sides of the triangle adjacent to the 90 degree angle are called the legs.
- The longest side of a right triangle is the hypotenuse.

Encourage students to say the Pythagorean theorem in words.

Point out that knowing the value of two variables in the Pythagorean theorem is sufficient for determining the third. See Example 4.

Suggest that students discard negative solutions obtained from radical equations in this lesson since the lengths of sides of triangles are nonnegative. See Example 5.

Specify how many decimal places are required of students when rounding.

Error Troubleshooting

General Tip: Remind students to set the value of c as the length of the hypotenuse; the values for a and b can be switched.

In *Review Question 18*, assume that the hypotenuse is also the diameter of the circle.

11.5 Distance and Midpoint Formulas

Learning Objectives

At the end of this lesson, students will be able to:

- Find the distance between two points in the coordinate plane.
- Find the missing coordinate of a point given the distance from another known point.
- Find the midpoint of a line segment.
- Solve real-world problems using distance and midpoint formulas.

Vocabulary

Terms introduced in this lesson:

distance
equidistant
midpoint

Teaching Strategies and Tips

Use Examples 1 and 2 to show how the Pythagorean Theorem is used to derive the distance formula.

- Teachers are encouraged to use a picture in the derivation.

Use Examples 3-5 and *Review Questions* 7, 8, and 15-20 as thinking problems.

- Draw pictures to help.
- Contrast these problems with the mechanical exercises of Example 2 and *Review Questions* 1-6.

Point out that because of the squares in the distance formula, the order in which the x -values (and the order of the y -values) are plugged in does not matter.

Point out that the midpoint of a segment is found by taking the average values of the x - and y -values of the endpoints.

In Example 9, suggest that students express their answers in radical form.

Error Troubleshooting

General Tip: For points with negative coordinates, remind students about the minus sign in the distance formula.

Example:

Find the distance between the two points.

$(-2, 5)$ and $(3, -8)$.

Hint: Plug the values of the two points into the distance formula; notice that parentheses were used around -8 .

$$d = \sqrt{(-2 - 3)^2 + (5 - (-8))^2}$$

11.6 Measures of Central Tendency and Dispersion

Learning Objectives

At the end of this lesson, students will be able to:

- Compare measures of central tendency.
- Measure the dispersion of a collection of data.
- Calculate and interpret measures of central tendency and dispersion for real-world situations.

Vocabulary

Terms introduced in this lesson:

measure of central tendency
average
data set
mean
median
mode
sum of values, number of values
ordered list
outlier
dispersion
variance
standard deviation

Teaching Strategies and Tips

Students learn that “average” has a general meaning:

- Averages describe the general characteristics of a group.
- Mean, median, and mode are examples of averages.

Use Examples 1 and 2 to introduce the mean.

Additional Examples:

Find the mean of the given data.

a. The students in Mr. Peterson’s math class took the AP Statistics exam. Their math scores are:

3, 2, 2, 3, 4, 4, 3, 1, 2, 3, 4, 5, 4, 2, 2, 3, 4, 3, 4, 5, 4, 3

Answer: 3.18

b. The weights, in ounces, of several cookies taken from the same package are:

0.95, 0.85, 0.93, 0.90, 0.97, 0.96, 0.87, 0.91

Answer: 0.92

c. The precipitation, in mm, for the city of Townville in the month of October is:

142.20, 0.02, 0.01, 12.15, 92.72, 103.21, 138.90, 102.92, 12.07, 1.00, 0.00, 0.00, 12.01, 21.02, 22.02, 87.91, 89.60, 132.72, 120.82,

Answer: 38.37

Use Examples 3 and 4 to introduce the median.

Additional Examples:

Find the median of the given data.

a. The students in Mr. Peterson's math class took the AP Statistics exam. Their math scores are:

3, 2, 2, 3, 4, 4, 3, 1, 2, 3, 4, 5, 4, 2, 2, 3, 4, 3, 4, 5, 4, 3

Answer: 3

b. The weights, in ounces, of several cookies taken from the same package are:

0.95, 0.85, 0.93, 0.90, 0.97, 0.96, 0.87, 0.91

Answer: 0.92

c. The precipitation, in mm, for the city of Townville in the month of October is:

142.20, 0.02, 0.01, 12.15, 92.72, 103.21, 138.90, 102.92, 12.07, 1.00, 0.00, 0.00, 12.01, 21.02, 22.02, 87.91, 89.60, 132.72, 120.82,

Answer: 12.185

Use Example 5 to introduce the mode.

Additional Examples:

Find the mode of the given data.

a. The students in Mr. Peterson's math class took the AP Statistics exam. Their math scores are:

3, 2, 2, 3, 4, 4, 3, 1, 2, 3, 4, 5, 4, 2, 2, 3, 4, 3, 4, 5, 4, 3

Answer: 3 and 4.

b. The weights, in ounces, of several cookies taken from the same package are:

0.95, 0.85, 0.93, 0.90, 0.97, 0.96, 0.87, 0.91

Answer: None.

c. The precipitation, in mm, for the city of Townville in the month of October is:

142.20, 0.02, 0.01, 12.15, 92.72, 103.21, 138.90, 102.92, 12.07, 1.00, 0.00, 0.00, 12.01, 21.02, 22.02, 87.91, 89.60, 132.72, 120.82,

Answer: 0.00

Use Examples 6-8 to introduce measures of dispersion.

Additional Examples:

Find the range, variance, and standard deviation of the given data.

a. The students in Mr. Peterson's math class took the AP Statistics exam. Their math scores are:

3, 2, 2, 3, 4, 4, 3, 1, 2, 3, 4, 5, 4, 2, 2, 3, 4, 3, 4, 5, 4, 3

Answer:

$$\text{range} = 4$$

$$s^2 = 1.109$$

$$s = 1.053$$

b. The weights, in ounces, of several cookies taken from the same package are:

0.95, 0.85, 0.93, 0.90, 0.97, 0.96, 0.87, 0.91

Answer:

$$\text{range} = 0.12$$

$$s^2 = 0.0018$$

$$s = 0.0430$$

c. The precipitation, in mm, for the city of Townville in the month of October is:

142.20, 0.02, 0.01, 12.15, 92.72, 103.21, 138.90, 102.92, 12.07, 1.00, 0.00, 0.00, 12.01, 21.02, 22.02, 87.91, 89.60, 132.72, 120.82,

Answer:

$$\text{range} = 142.2$$

$$s^2 = 2601$$

$$s = 51$$

In Example 7, students learn to separate the components of the standard deviation formula into manageable pieces.

- Suggest that students reconstruct such a multi-column table for *Review Question 2*.
- Suggest that students use a calculator to only to check their calculations.

Encourage students to provide the kind of explanations shown in Example 9 for *Review Questions 4* and *5*. Ask:

- What does a larger mean imply?
- What does a larger standard deviation imply?

Emphasize that the appropriateness of an average depends on the shape of the distribution of the data.

- Use the mean for tightly clustered data sets.
- Use the median for data sets with a lot of spread.
- Use the mode when there is no apparent pattern in the distribution.

Error Troubleshooting

General Tip: Remind students to order the data set when computing the median and the range.

General Tip: Suggest that students transfer data to calculator *carefully*.

- Have students put a line through the data points as they get transferred.
- Have students count the number of data points in both lists to prevent omissions.

11.7 Stem-and-Leaf Plots and Histograms

Learning Objectives

At the end of this lesson, students will be able to:

- Make and interpret stem-and-leaf plots.
- Make and interpret histograms.
- Make histograms using a graphing calculator.

Vocabulary

Terms introduced in this lesson:

visual representation of data
stem-and-leaf plot

stem, leaf
ordered stem-and-leaf plot

frequency
histogram
bins
continuous data
round
truncate

Teaching Strategies and Tips

In the previous lesson, students learned that the appropriateness of an average depends on the shape of the distribution of the data.

- Plotting data is essential and should come naturally as a first step in data analysis.
- In this lesson, students learn to group and visualize data using stem-and-leaf plots and histograms. See Examples 1 and 2, respectively.

Emphasize the similarity between histograms and stem-and-leaf plots – a stem-and-leaf plot resembles a histogram on its side.

Teachers are encouraged to present several examples of stem-and-leaf plots consisting of different kinds of data sets, such as: clustered data that comes as three-digit numbers, data spread out coming in three-digits, data consisting of numbers with 1 or two decimal places, and data sets with decimals between 0 and 1.

Additional Examples:

Arrange the data into a stem-and-leaf plot.

a. The students in Mr. Peterson's math class took the AP Statistics exam. Their math scores are:

3, 2, 2, 3, 4, 4, 3, 1, 2, 3, 4, 5, 4, 2, 2, 3, 4, 3, 4, 5, 4, 3

b. The weights, in ounces, of several cookies taken from the same package are:

0.95, 0.85, 0.93, 0.90, 0.97, 0.96, 0.87, 0.91

c. The precipitation, in mm, for the city of Townville in the month of October is:

142.20, 0.02, 0.01, 12.15, 92.72, 103.21, 138.90, 102.92, 12.07, 1.00, 0.00, 0.00, 12.01, 21.02, 22.02, 87.91, 89.60, 132.72, 120.82,

Teachers are encouraged to show side-by-side histograms for the same data set as patterns can be more or less apparent with different number of bins.

- By increasing or decreasing bin-widths, students see how the shape of the distribution changes for better or worse.
- Some bin-widths can be random-looking or even misleading.

Error Troubleshooting

Remind students in *Review Questions* 1 and 4 to construct *ordered* stem-and-leaf plots, placing the leaves on each branch in ascending order, before correctly determining the median and the mode.

11.8 Box-and-Whisker Plots

Learning Objectives

At the end of this lesson, students will be able to:

- Make and interpret box-and-whisker plots.
- Analyze effects of outliers.
- Make box-and-whisker plots using a graphing calculator.

Vocabulary

Terms introduced in this lesson:

first quartile, third quartile

five number summary

whiskers

inter-quartile range (IQR)

range

raw data

ordered list

outlier, mild outlier, extreme outlier

Teaching Strategies and Tips

Use the introduction to point out that the median can be used to divide a data set into four quarters. See also Examples 1 and 2.

- After finding the quartiles, it is possible to construct the five-number summary and corresponding box-and-whisker plot.
- After finding the quartiles, it is possible to calculate the IQR.

Emphasize interpreting the box plot:

- 50% of the data set lies between the first and third quartiles (IQR).
- 75% of the data set lies above the first quartile; 75% of the data set lies below the third quartile.
- The range is the distance from one whisker to the other.
- Compare the relative size of the box to the length of the whiskers: short whiskers indicate clustered data; long whiskers indicate a spread-out data set.

- If one whisker is shorter than another, then the distribution is skewed.

Construct box-and-whisker plots for two data sets and compare them side-by-side. Point out that this makes drawing inferences easy. See Example 3.

Compare the range and IQR for a data set. Ask:

- For what kind of distributions should the IQR be used? the range?

Additional Example:

For the data set below, calculate the range and IQR. Which measure of dispersion do you think will give a better indication of the spread in the data?

2, 5, 8, 8, 8, 9, 9, 10, 100

Error Troubleshooting

NONE.

CHAPTER 12 TE Rational Equations and Functions; Topics in Statistics

CHAPTER OUTLINE

12.1 INVERSE VARIATION MODELS

12.2 GRAPHS OF RATIONAL FUNCTIONS

12.3 DIVISION OF POLYNOMIALS

12.4 RATIONAL EXPRESSIONS

12.5 MULTIPLICATION AND DIVISION OF RATIONAL EXPRESSIONS

12.6 ADDITION AND SUBTRACTION OF RATIONAL EXPRESSIONS

12.7 SOLUTIONS OF RATIONAL EQUATIONS

12.8 SURVEYS AND SAMPLES

Overview

In this chapter, students are introduced to inverse variation. They graph rational functions and divide polynomials. After being introduced to rational expressions they learn to add, subtract, multiply, and divide them. Students then solve rational equations. The chapter ends with surveys and sampling methods.

Suggested pacing:

Inverse Variation Models - 1 hr Graphs of Rational Functions - 1 – 2 hrs Division of Polynomials - 1 hr Rational Expressions - 1 hr Multiplication and Division of Rational Expressions - 1 hr Addition and Subtraction of Rational Expressions - 1 – 2 hrs Solutions of Rational Equations - 1 hr Surveys and Samples - 1 hr

If you would like access to the Solution Key FlexBook for even-numbered exercises, the Assessment FlexBook and the Assessment Answers FlexBook please contact us at teacher-requests@ck12.org.

Problem-Solving Strand for Mathematics

The first portion of this chapter cements connections between algebraic and geometric ways (graphical displays) of representing functions and relationships in mathematics. Vertical, horizontal, and oblique asymptotes visually confirm what students have learned earlier in their studies: division by zero is undefined in our number system. The connection between inverse variation models and the graphs of rational functions is presented in the examples and review questions which pose real-world problems using rational functions.

In the mid-lesson of this unit, the division of polynomials is related back once again to graphing and the horizontal asymptote studied previously. In the lesson, *Rational Expressions*, values that are excluded when simplifying rational expressions are shown to be those very values that are vertical asymptotes, values that cannot exist for x . Students who become adept at moving between the algebraic and the graphic will have more insights into problems they encounter and thus more tools for tackling future challenges.

The last lesson of the unit looks closely at topics in statistics. Students are asked to identify biased questions as well as biased sample populations and to display, analyze, and interpret statistical data effectively. Let students know they will have a survey project to complete before introducing the Surveys and Samples materials lesson and before reviewing Examples 3 and 4, *Designing a Survey*, and Examples 5 and 6, *Display, Analyze, and Interpret Data*, since the information needs to be seen in context in order for its value to be recognized.

Alignment with the NCTM Process Standards

The first lessons of this chapter consistently recognize and use connections among mathematical ideas (CON.1), encourage students to understand how mathematical ideas interconnect and build on one another to produce a coherent whole (CON.2), and—especially in real-world problem solving scenarios—recognize and apply mathematics in contexts outside of mathematics (CON.3). In the lessons, Rational Expressions and Solutions of Rational Equations, classic work and motion problems are presented so as to put these problem-solving techniques into meaningful perspective (PS.1, PS.2, and PS.3). In the Surveys and Samples lesson, students create and use representations to organize, record, and communicate mathematical ideas (R.1); select, apply, and translate among mathematical representations to solve problems (R.2); and use representations to model and interpret physical, social, and mathematical phenomena (R.3).

- CON.1 - Recognize and use connections among mathematical ideas.
- CON.2 - Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
- CON.3 - Recognize and apply mathematics in contexts outside of mathematics.
- PS.1 - Build new mathematical knowledge through problem solving.
- PS.2 - Solve problems that arise in mathematics and in other contexts.
- PS.3 - Apply and adapt a variety of appropriate strategies to solve problems.
- R.1 - Create and use representations to organize, record, and communicate mathematical ideas.
- R.2 - Select, apply, and translate among mathematical representations to solve problems.
- R.3 - Use representations to model and interpret physical, social, and mathematical phenomena.

12.1 Inverse Variation Models

Learning Objectives

At the end of this lesson, students will be able to:

- Distinguish direct and inverse variation.
- Graph inverse variation equations.
- Write inverse variation equations.
- Solve real-world problems using inverse variation equations.

Vocabulary

Terms introduced in this lesson:

variation

direct variation

inverse variation

joint variation

constant of proportionality

increase, decrease

Teaching Strategies and Tips

This lesson focuses on inverse variation models and graphing inverse variation equations. Use it to motivate rational functions, which are covered in the next six lessons.

Remind students about having learned direct variation in chapter *Graphs of Equations and Functions*.

- Point out that direct variation is a linear relationship.
- The x and y -intercepts are 0.
- The slope of the line is the only parameter, denoted by k , and called the constant of proportionality.
- It takes only one more point to determine the direct variation.

Some examples of direct variation relationships are:

- Height of a person and the length of their shadow on flat ground.
- Circumference and radius of the circle.
- Weight of an object on a spring and the amount the spring has stretched.

In the examples and *Review Questions*, have students decide on a variation model first and then solve for the constant of proportionality using the given information. This determines the equation of the variation which is necessary for answering the rest of the problem.

Use Example 1 to illustrate the graph of an inverse variation.

- Construct a similar table of values. Allow students to observe the function's behavior numerically.

Remind students of scientific notation in Example 6.

In applied problems such as Examples 5 and 6, emphasize that direct variations are ubiquitous and significant in the real-world.

In *Review Questions* 1-4, encourage students to apply stretches to the basic graph $y = \frac{1}{x}$.

Example:

Graph the following inverse variation relationship.

$$y = \frac{10}{x}.$$

Hint: Since $y = \frac{10}{x} = 10 \cdot \frac{1}{x}$, the graph can be obtained from that of $y = \frac{1}{x}$ by stretching by a factor of 10.

Error Troubleshooting

Remind students in Example 1 that dividing by zero is undefined.

Remind students in Example 6 to square the 5.3 which is in parentheses:

$$K = 740(5.3 \times 10^{-11})^2 = 740 \cdot 5.3^2 \cdot 10^{-22}$$

12.2 Graphs of Rational Functions

Learning Objectives

At the end of this lesson, students will be able to:

- Compare graphs of inverse variation equations.
- Graph rational functions.
- Solve real-world problems using rational functions.

Vocabulary

Terms introduced in this lesson:

rational function
horizontal asymptote, vertical asymptote
oblique (slant) asymptote

Teaching Strategies and Tips

Reconstruct the tables in Examples 2-4 to remind students of the inverse relationship.

Explore several rational functions side-by-side.

- Have students make note of the degrees of the numerator and denominator and any horizontal and vertical asymptotes.
- Point out that what sets rational functions apart from other functions in this course is *division*.
- Division creates the asymptotes and branches.
- Remind students that dividing by zero is undefined and is denoted on the graph by a vertical dashed line.

Asymptotes are denoted by dashed lines. Remind students that asymptotes are not part of the function and only serve to show how the graph approaches certain values.

- Point out that graphing calculators may display asymptotes using a solid line.

Have students rewrite the steps for finding asymptotes preceding Example 5 for themselves.

Encourage graphing rational functions by hand. Use a graphing calculator only as a way to check.

- Sketching graphs can solidify student understanding of x -intercepts, y -intercepts, factoring, and domains.

Error Troubleshooting

In Examples 2-4, have students choose enough values for their tables to determine the behavior of the function accurately. Remind them to pick values close to the vertical asymptotes.

12.3 Division of Polynomials

Learning Objectives

At the end of this lesson, students will be able to:

- Divide a polynomial by a monomial.
- Divide a polynomial by a binomial.
- Rewrite and graph rational functions.

Vocabulary

Terms introduced in this lesson:

rational expression
numerator
denominator
common denominator
dividend
divisor
quotient
remainder

Teaching Strategies and Tips

Students learned in chapter *Factoring Polynomials* how to add, subtract, and multiply polynomials. This lesson completes that discussion with dividing polynomials.

- Emphasize that the quotient of two polynomials forms a rational expression which is studied in its own right (rational functions).

Use Example 1 to demonstrate dividing a polynomial by a monomial.

- Remind students that *each* term in the numerator must be divided by the monomial in the denominator. See Example 2.

Use Example 3 to motivate long division of polynomials.

- To write the answer, remind students that:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

- To check an answer, have students use the equivalent form:

$$\text{dividend} = (\text{divisor} \times \text{quotient}) + \text{remainder}$$

Have students rewrite for themselves the four cases for graphing rational functions preceding Example 5.

Error Troubleshooting

General Tip: Students often incorrectly cancel a factor not common to all the terms.

- Example:

$$\frac{\cancel{a}x + b}{\cancel{a}y} \neq \frac{x + b}{y}$$

- When students cancel the a above, they violate order of operations. Remind students that the fraction sign is a grouping symbol (parentheses) and therefore the numerator and denominator must be simplified before dividing.
- Otherwise, if the numerator and denominator are *completely factored*, then the order of operations says to multiply or divide; therefore, canceling is justified.
- Have students write out the step preceding the canceling:

Example:

$$\frac{ax + ab}{ay} = \frac{a(x + b)}{ay}$$

Then canceling is apparent:

$$\frac{ax + ab}{ay} = \frac{a(x + b)}{ay} = \frac{\cancel{a}(x + b)}{\cancel{a}y} = \frac{x + b}{y}$$

- Other common cancelling errors are:

a. $\frac{\cancel{a}x + ab}{\cancel{a}y} \neq \frac{x + ab}{y}$ (forgetting to remove the canceled factor)

b. $\frac{\cancel{a}}{x + \cancel{a}} \neq \frac{1}{x}$

12.4 Rational Expressions

Learning Objectives

At the end of this lesson, students will be able to:

- Simplify rational expressions.
- Find excluded values of rational expressions.
- Simplify rational models of real-world situations.

Vocabulary

Terms introduced in this lesson:

lowest terms
canceling common factors
common terms
excluded value
removable zero

Teaching Strategies and Tips

In this lesson, students learn what removable zeros are, how to find them, how they are related to excluded values, and what they look like graphically.

- Point out that removable zeros are “divisions by zero” removed by simplifying the rational expression.
- See Example 2.

Have students review factoring.

Additional Examples:

a. $\frac{16x^2+12x}{4x} = \frac{4x(4x+3)}{4x} = 4x + 3, (x \neq 0)$

b. $\frac{56x^3+14x^2}{4x^2+5x+1} = \frac{14x^2(4x+1)}{(x+1)(4x+1)} = \frac{14x^2}{x+1}, (x \neq -\frac{1}{4})$

Optional: In some rational expressions, it may appear to students like nothing will cancel at first, despite the similar terms. Suggest that students try factoring out a negative and then rearranging the order of the terms.

c. $\frac{1-x}{x-1} = \frac{-(x-1)}{x-1} = -1, x \neq 1$

When factoring out the minus sign, suggest that students use brackets to keep the negative outside and so that it will be harder to lose it.

$$d. \frac{1-x^2}{x^2+x-2} = \frac{(1-x)(1+x)}{(x-1)(x+2)} = - \left[\frac{(x-1)(x+1)}{(x-1)(x+2)} \right] = -\frac{x+1}{x+2}, x \neq 1$$

Point out that the rules for working with rational expressions are the same as those for working with ordinary fractions.

- *Simplifying* a rational expression means the same as simplifying a fraction – that the numerator and denominator of the rational expression have no common factors.
- Remind students after canceling all the terms of a numerator (or denominator) that a factor of 1 remains.

In Example 3, have students use scientific notation.

Error Troubleshooting

General Tip: Suggest that students check their work after simplifying a rational expression by substituting the variable with a number.

General Tip: Remind students that in order to cancel a factor, it must be common to the entire numerator and denominator.

12.5 Multiplication and Division of Rational Expressions

Learning Objectives

At the end of this lesson, students will be able to:

- Multiply rational expressions involving monomials.
- Multiply rational expressions involving polynomials.
- Multiply a rational expression by a polynomial.
- Divide rational expressions involving polynomials.
- Divide a rational expression by a polynomial.
- Solve real-world problems involving multiplication and division of rational expressions.

Vocabulary

NONE

Teaching Strategies and Tips

Use the ordinary fractions in Example 1 to motivate multiplying and dividing variable rational expressions.

Use Example 2 to point out that an answer can be more readily obtained by reducing common factors *before* multiplying.

- Suggest students to factor all polynomial expressions as much as possible first.

Error Troubleshooting

In *Review Questions* 9, 12, and 16, remind students to write the expression on the right as a fraction with denominator equal to 1, and then proceed to multiply or divide.

General Tip: Remind students to rewrite division problems as multiplication problems.

General Tip: In a division problem, some students will invert the first fraction. Remind students to invert the second.

12.6 Addition and Subtraction of Rational Expressions

Learning Objectives

At the end of this lesson, students will be able to:

- Add and subtract rational expressions with the same denominator.
- Find the least common denominator of rational expressions.
- Add and subtract rational expressions with different denominators.
- Solve real-world problems involving addition and subtraction of rational expressions.

Vocabulary

Terms introduced in this lesson:

least common denominator (LCD)

least common multiple (LCM)

prime factorization

factor completely

equivalent fraction

Teaching Strategies and Tips

Use the ordinary fractions in Examples 1 and 5 to motivate adding and subtracting variable rational expressions with and without common denominators, respectively.

Draw the analogy between finding the LCM of polynomials and the LCM of integers.

- Use *prime factorization* of numbers and polynomials.
- In general, the LCM is found by taking each factor to the highest power that it appears in each expression.

In Examples 6, 8, and 9, remind students to distribute the minus to each term in the second rational expression. See also *Review Questions* 4, 5, 8, 17, 18, 21, 23, 27, and 30.

Draw the analogy between the formula: *part of the task completed = rate of work · time spent on the task* and the formula: *distance = rate · time*.

In *Review Questions* 34-36, encourage students to set up a table similar to the one in Example 12.

- Emphasize that all known and unknown variables for each person or machine can be listed, which makes organizing the given information easy.

- Combining parts of the task completed by each person or machine is a matter of reading across the rows or down the columns depending on how the table is set up.
- Many students find tables useful for work problems; others rely heavily upon it.

In *Review Questions* 34-36,

- Suggest that students begin by looking at the part of the task completed by each person or machine *separately*.
- Encourage students to check the reasonableness of their answers. Ask: What kind of answer should we expect based upon the given information?

Error Troubleshooting

In *Review Questions* 7 and 8, suggest that students factor out a negative from the second rational expression first.

- In Example 7, the LCM of $x - 4$ and $4 - x = -(x - 4)$ is $x - 4$.

General Tip: Remind students to find the LCD of rational expressions by factoring. Students needlessly use larger common multiples when expressions are not completely factored.

General Tip: Have students leave the LCM in factored form. This makes simplifying and determining excluded values easier.

12.7 Solutions of Rational Equations

Learning Objectives

At the end of this lesson, students will be able to:

- Solve rational equations using cross products.
- Solve rational equations using lowest common denominators.
- Solve real-world problems with rational equations.

Vocabulary

Terms introduced in this lesson:

rational equation
cross products

Teaching Strategies and Tips

Students should now be able to solve linear, quadratic, and radical equations; and be able to graph linear, quadratic, exponential, and radical functions. In this lesson, rational equations and functions are added to these lists.

Emphasize that the first step in solving rational equations is eliminating the denominators on all the terms.

- Emphasize that the last step is checking their answers in the original equation.

Use Example 1 to remind students about cross-multiplication.

- Suggest that students simplify the terms of the equation and move them to one side before cross-multiplying.

Use Example 4 to motivate using the LCD strategy.

- Students learn to eliminate denominators by multiplying all terms by the LCD.
- Emphasize that this method is preferred over cross-multiplication when there are two or more terms in an equation.
- Have students completely factor each expression first.

After multiplying both sides by the LCD, have students use colors to cancel like factors. This helps students track their canceling and prevents canceling a factor more than once.

Reconstruct the tables in Examples 7-9 as a useful way to display the given information in *Review Questions* 19-24.

Error Troubleshooting

General Tip: Remind students to check their answers in the *original* equation.

- Emphasize that this is a necessary step for rational equations since the variable appears in denominators.
- Essentially, if an answer makes any denominator zero, then that value is not a solution to the equation.

12.8 Surveys and Samples

Learning Objectives

At the end of this lesson, students will be able to:

- Classify sampling methods.
- Identify biased samples.
- Identify biased questions.
- Design and conduct a survey.
- Display, analyze, and interpret statistical survey data.

Vocabulary

Terms introduced in this lesson:

census
sampling method
representative sample
random sampling
stratified sampling
sample size
bias
biased sample
population
sample
cherry picking
non-response bias
self-selected respondents
poll
survey
biased question
question order
design and conduct a survey
face-to-face interview
self-administered survey
survey data

Teaching Strategies and Tips

Students learn about surveys and samples as ways of collecting information:

- Examples accompany each type of method and help students put them into context.
- Emphasize the advantages and disadvantages of each type of sampling method: census, random, and stratified.

Additional Examples:

Identify the type of sampling used.

a. A superintendent has a computer generate a list of 100 teachers from the 10 schools in her district and interviews them about working conditions.

Answer: Random sample

b. At a certain high school there are 730 freshmen, 512 sophomores, 475 juniors, and 103 seniors. A reporter from the school newspaper interviews 15 freshmen, 12 sophomores, 13 juniors, and 10 seniors about their college plans.

Answer: Stratified sampling

c. Around mid-semester, a certain university requires a teacher's evaluation. A form is given to each student in every class to fill out.

Answer: Census

Emphasize that samples consisting of over or under-represented sub-groups are biased.

- Point out that a biased sample does not accurately reflect the spread of people present in the population and, therefore, the sample should not be expected to represent the entire population.

Additional Examples:

Identify each sample as biased or unbiased. If the sample is biased explain how you would improve your sampling method.

a. County health officials randomly selected non-smokers walking out of a bar. They politely asked every other person whether or not second-hand smoke affects them.

Hint: Possible under-represented group; people who dislike second-hand smoke tend to stay away from bars.

b. At a popular ski resort, people were surveyed about the cafeteria food. The survey was handed out in the cafeteria and was to be mailed in, postage paid. The results were to be published the county newspaper.

Hint: Possible bias includes the voluntary response of the people being surveyed. For example, those who find the time and effort to mail in the survey may be predisposed to a certain opinion.

c. A potato chip manufacturer packaging bags of chips maintains quality control by randomly selecting 100 bags over the course of 48 hours and weighs each bag. They then inspect 10 bags by opening them and tasting the chips.

Hint: No apparent bias.

d. People visiting the website of a popular teen magazine had the option of participating in a poll on whether the legal drinking age should be lowered.

Hint: Possible bias includes the voluntary response of the people being surveyed.

Teachers are encouraged to read and discuss with their students the ways to spot biased questions. See the list following Example 2.

Additional Examples:

Examine the question for possible bias. If you think the question is biased, indicate how to propose a better question.

a. Given that underage drinking is responsible for 17% of all car accidents, do you think the legal drinking age should be lowered?

Hint: Biased. Possible alternative: "Do you think lowering the legal drinking age is appropriate?"

12.8. SURVEYS AND SAMPLES

b. Should corporations that use diesel fuel in their transports, and thus pollute the environment, pay an environmental tax to help clean the air?

Hint: Biased. Possible alternative: “Should corporations using diesel fuel pay an environmental tax?”

c. Are you in favor of continued funding for the high school’s remarkable theater program?

Hint: Biased. Possible alternative: “Do you favor continued funding for the high school theater program?”

d. Do you think all high school students should be required to take a physical education course?

Hint: Likely unbiased.

Use Examples 3 and 4 to demonstrate step-by-step how to conduct a survey.

Use the last part of the lesson to show students how to display, analyze, and interpret survey data. Appropriate ways to display data are: pie-charts and bar-graphs for categorical data; stem-and-leaf plots, histograms, and box-and-whisker plots for numerical data.

Error Troubleshooting

NONE

